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CMMB
Centre for Mathematical Medicine and Biology

Networks of nonsmooth dynamical systems

Yi Ming Lai

EPSRC
Engineering and Physical Sciences Research Council

The Master Stability Function (MSF)

- Useful approach for evaluating stability of networks of dynamical systems
- Start with a synchronization manifold
- (Guaranteed to exist for constant row sum coupling matrix)
- Construct linearized variational equation around synchronous state
- Block diagonalize according to the eigenvectors of the coupling matrix
- The full system is now reduced to individual problems for each eigenvalue of the coupling matrix

The Master Stability Function (MSF)

L M Pecora and T L Carroll. Master stability functions for synchronized coupled systems. Physical Review Letters, 80:2109–2112, 1998.

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} w_{ij} \left[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)\right]$$

$$\equiv \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{G}_{ij} \mathbf{H}(\mathbf{x}_j)$$

$$\mathbf{x}_i, \mathbf{F}, \mathbf{H} \in \mathbb{R}^m$$

$$i = 1, \ldots, N$$

Graph Laplacian

$$\mathcal{G}_{ij} = -w_{ij} + \delta_{ij} \sum_k w_{ik}$$

Synchronisation manifold

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \ldots = \mathbf{x}_N(t) = \mathbf{s}(t) \qquad \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$$

The Master Stability Function (MSF)

Variational problem $\mathbf{x}_i(t) = \mathbf{s}(t) + \delta\mathbf{x}_i(t)$

$$\frac{d}{dt}\delta\mathbf{x}_i = D\mathbf{F}(\mathbf{s})\delta\mathbf{x}_i - \sigma D\mathbf{H}(\mathbf{s})\sum_{j=1}^{N}\mathcal{G}_{ij}\delta\mathbf{x}_j$$

Nice notation $\mathbf{U} = (\delta\mathbf{x}_1, \ldots, \delta\mathbf{x}_N) \in \mathbb{R}^{N\times m}$

$$\dot{\mathbf{U}} = (I_N \otimes D\mathbf{F}(\mathbf{s}))\,\mathbf{U} - \sigma\,(\mathcal{G} \otimes D\mathbf{H}(\mathbf{s}))\,\mathbf{U}$$

Block diagonalise using

$$\mathcal{G}P = P\Lambda$$

$$\mathbf{V} = (P \otimes I_m)^{-1}\mathbf{U} \qquad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$$

$$\dot{\mathbf{V}} = (I_N \otimes D\mathbf{F}(\mathbf{s}))\,\mathbf{V} - \sigma\,(\Lambda \otimes D\mathbf{H}(\mathbf{s}))\,\mathbf{V}$$

The Master Stability Function (MSF)

N-block structure with the dynamics in each block, indexed by $l = 1, \ldots, N$:

$$\xi_l \in \mathbb{C}^m$$

$$\dot{\xi}_l = [\mathbf{DF}(\mathbf{s}) - \beta_l \mathbf{DH}(\mathbf{s})]\, \xi_l$$

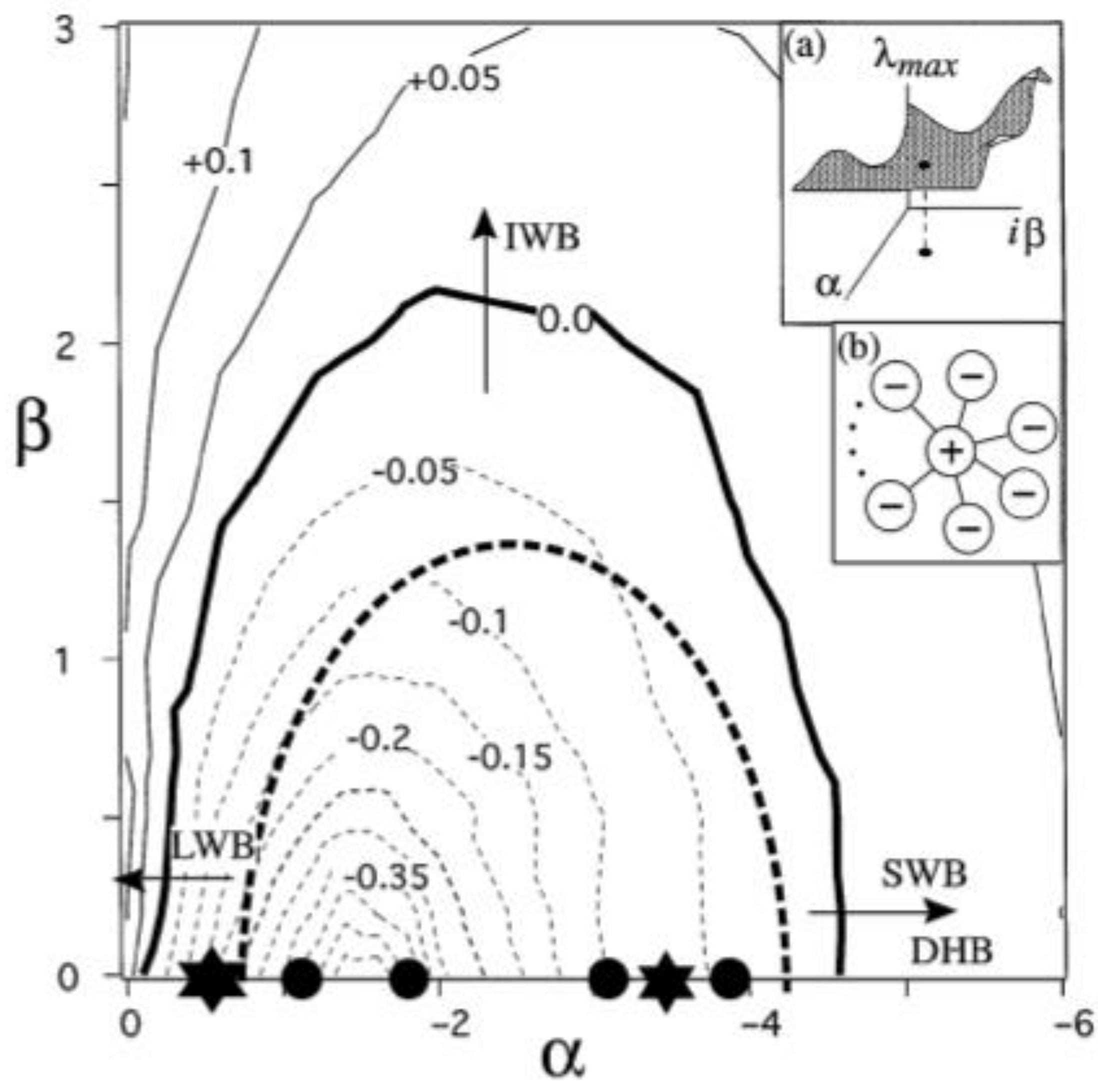
$$\beta_l = \sigma\lambda_l \in \mathbb{C}$$

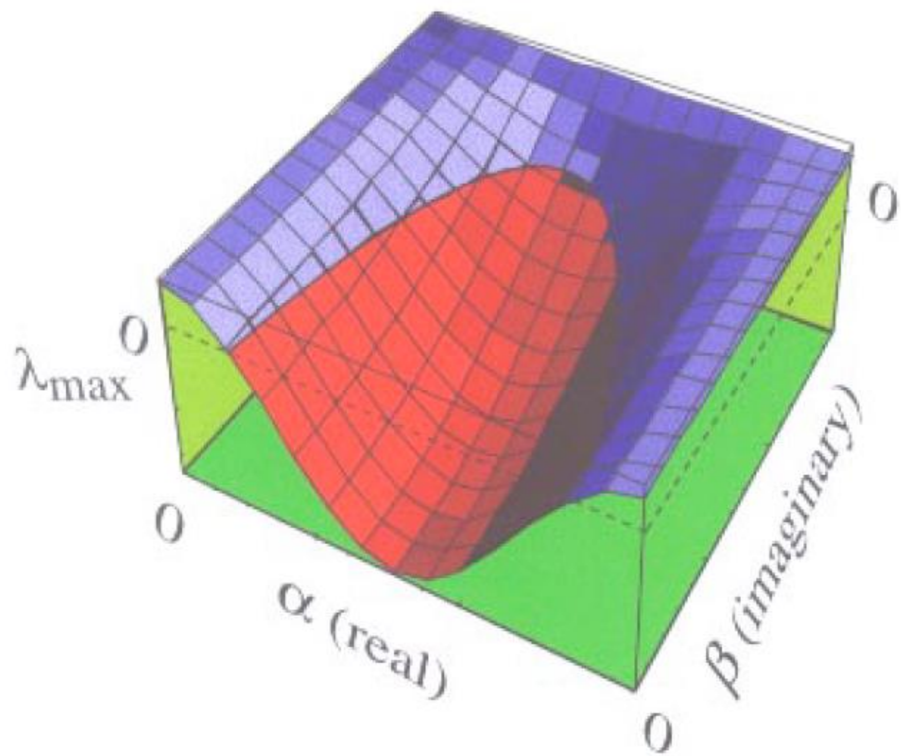
The **MSF** is defined as the function which maps the complex number β to the greatest Floquet exponent of the variational equation. The synchronous state of the system of coupled oscillators is stable if the MSF is negative at $\beta = \sigma\lambda_l$ where λ_l ranges over the eigenvalues of the matrix $\mathcal{G}$ (excluding $\lambda_1 = 0$).

- We can reverse this definition to evaluate the stability of synchrony of the system in terms of a single complex parameter

- Stability determined for any network by simply seeing where the eigenvalues lie in the complex plane

MSF for General Networks



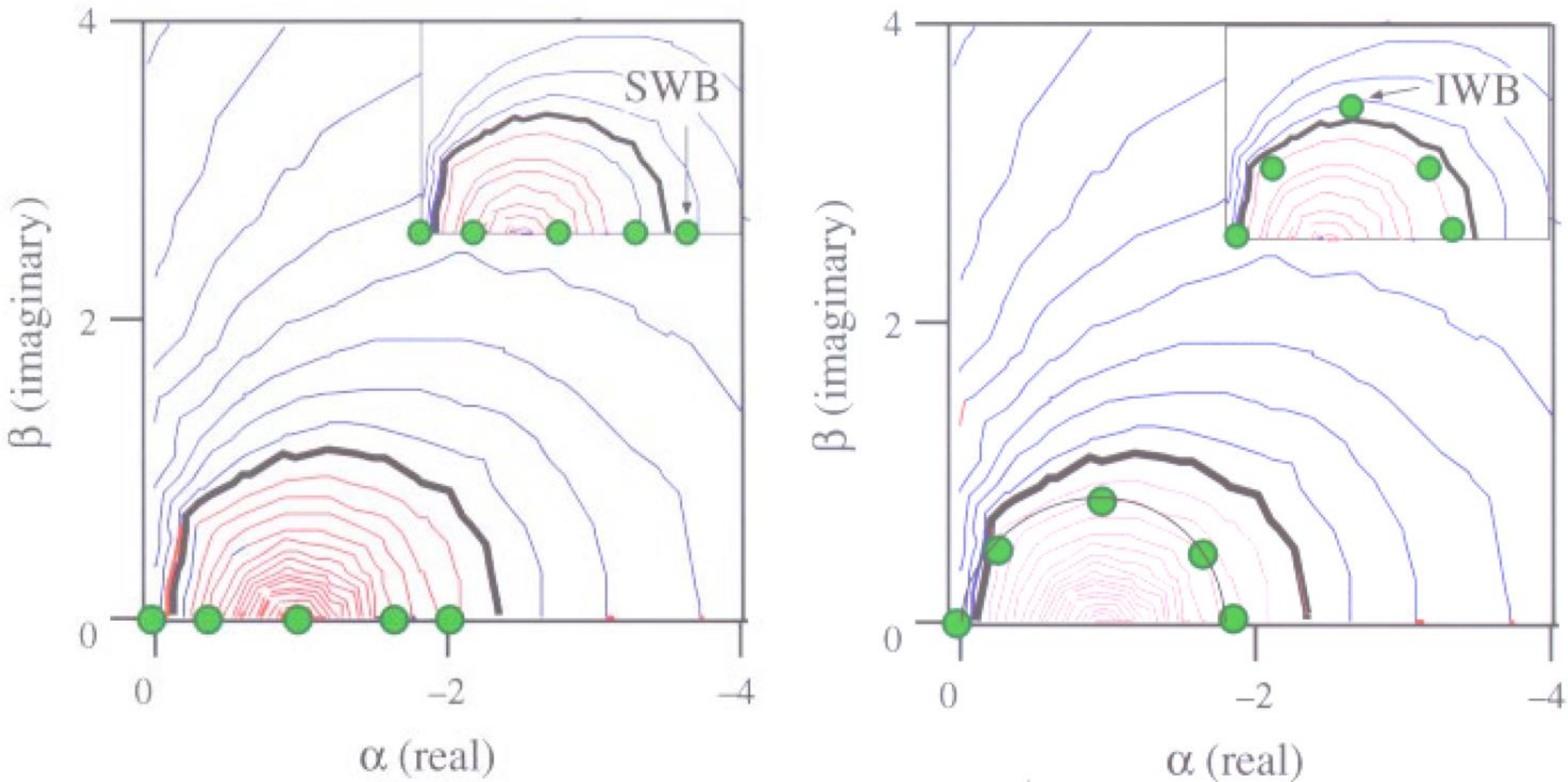


MSF for coupled Rössler oscillators

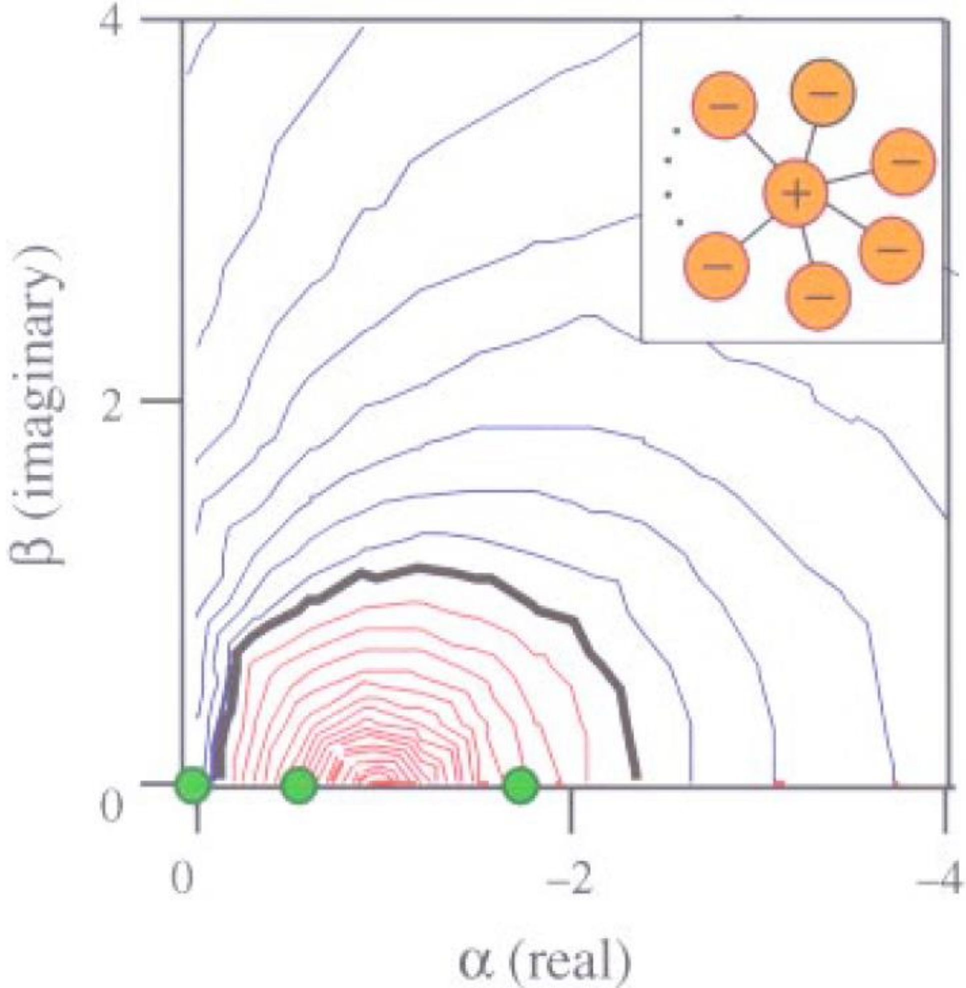
Figure from Pecora & Carroll PRL 1998, Pecora et al. Int J Bif Chaos 2000

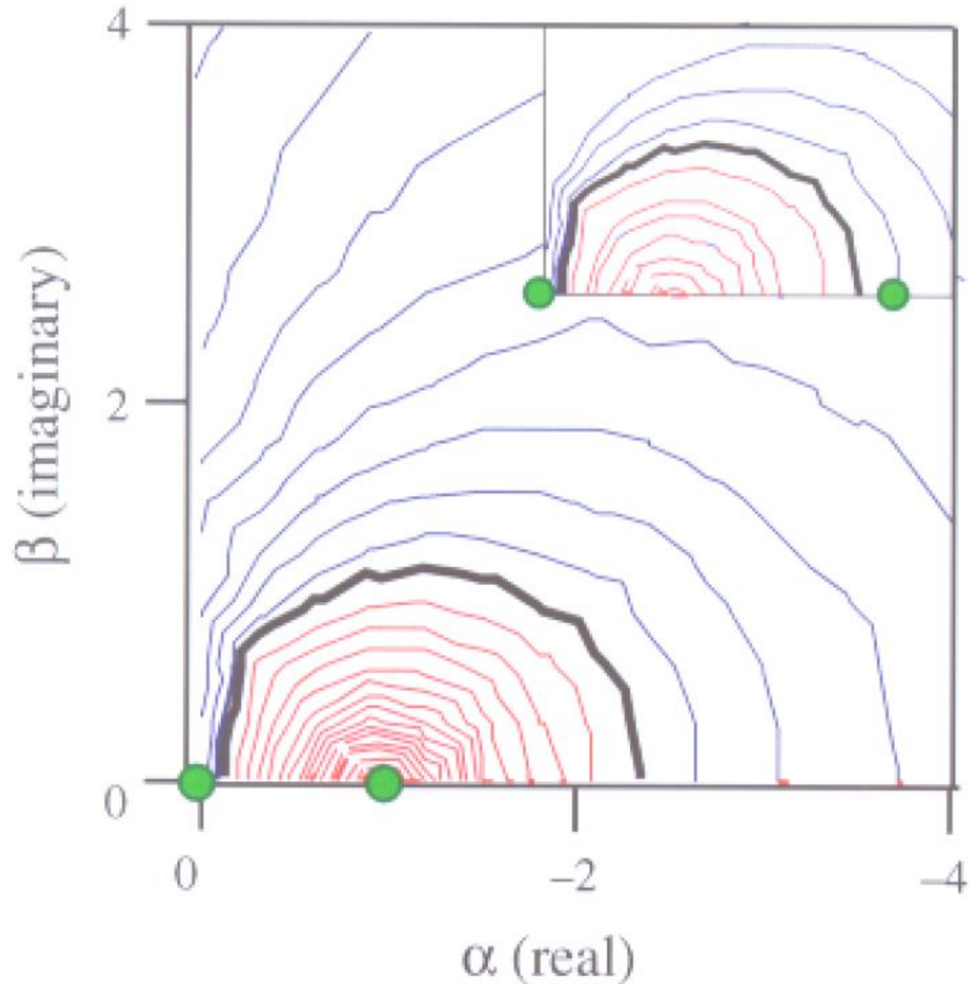
MSF for General Networks

We can investigate different bifurcations and spatial patterns by looking at the corresponding eigenvector to the eigenvalue leaving the region of stability

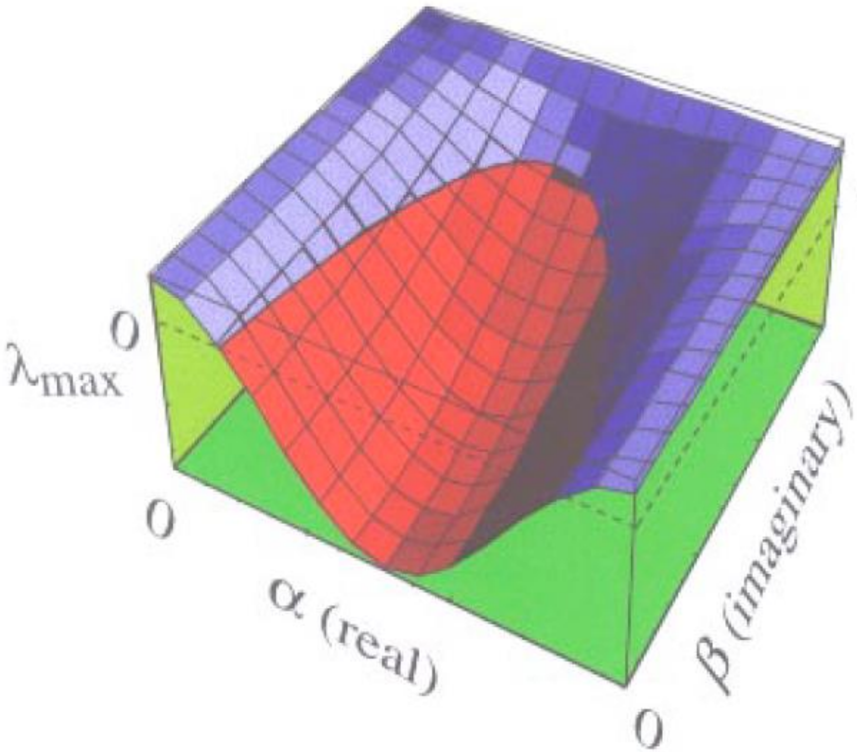


MSF for General Networks

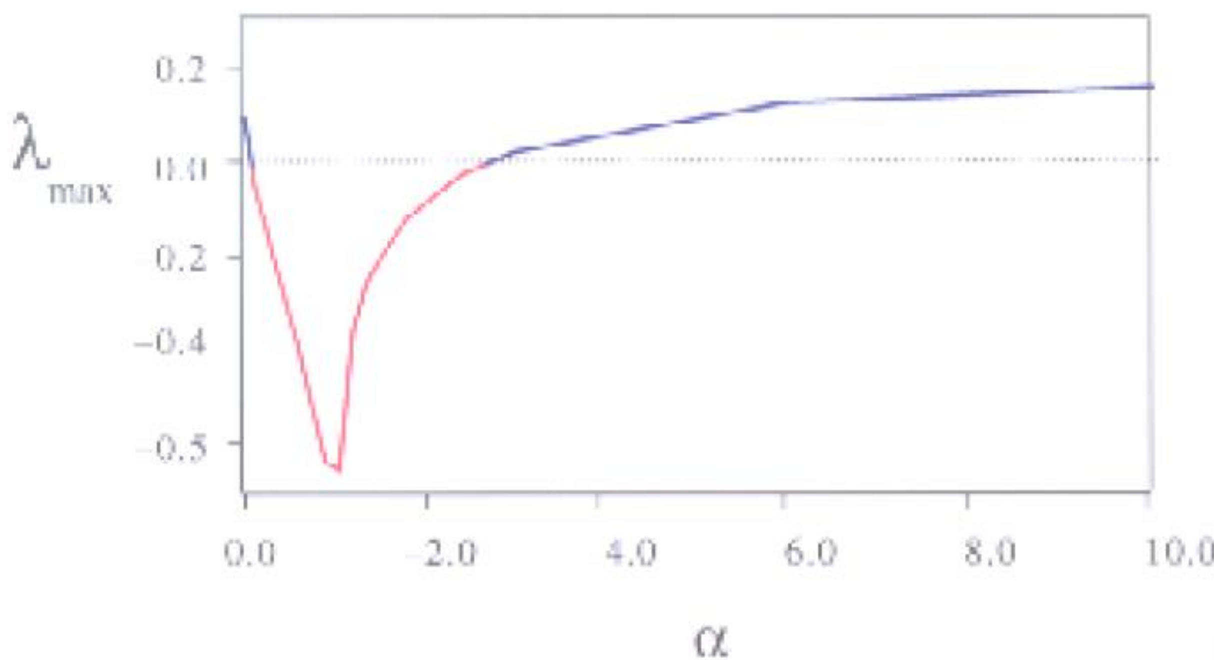




MSF for General Networks

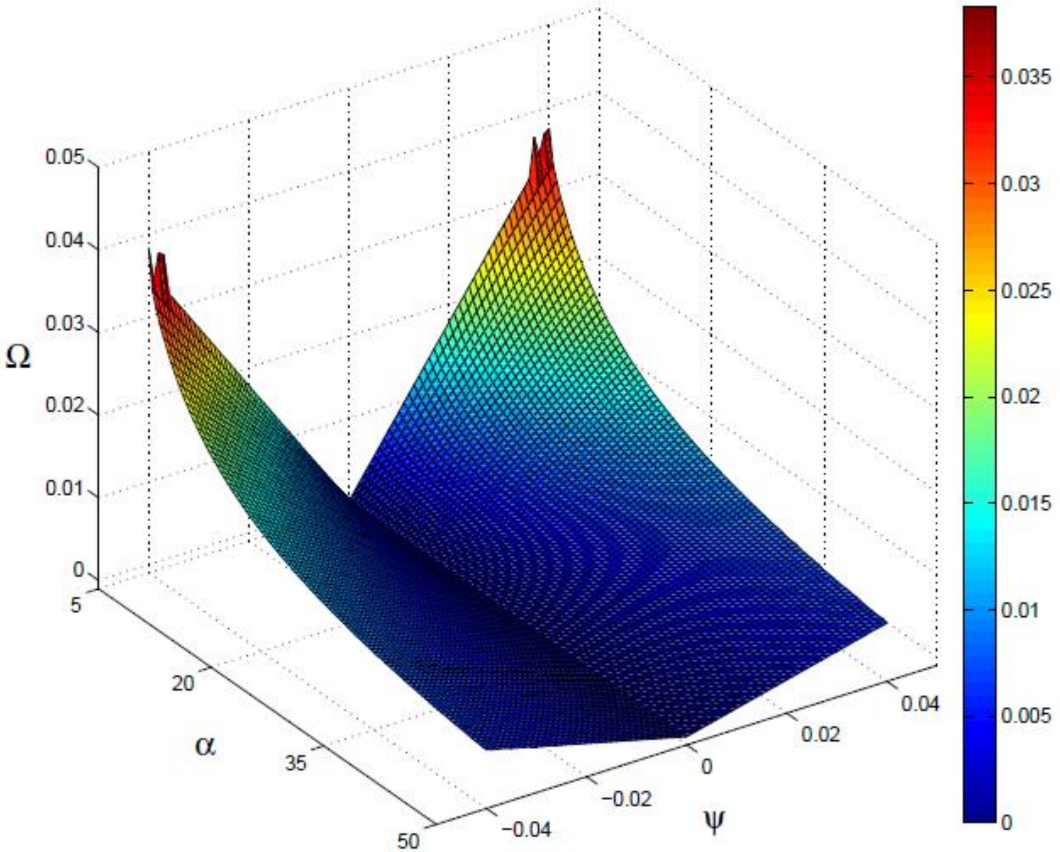


If the coupling matrix is symmetric, eigenvalues live on the real line, allowing us to get a collapsed view of the MSF



Extensions of the MSF approach

Near identical oscillators



$$\dot{\xi} = \left[D_w f - \alpha \cdot DH \right] + D_\mu f \cdot \psi$$

Parameter “mismatch”

Sun, Bollt, Nishikawa EPL 2009

Extensions of the MSF approach

Homogeneous delay

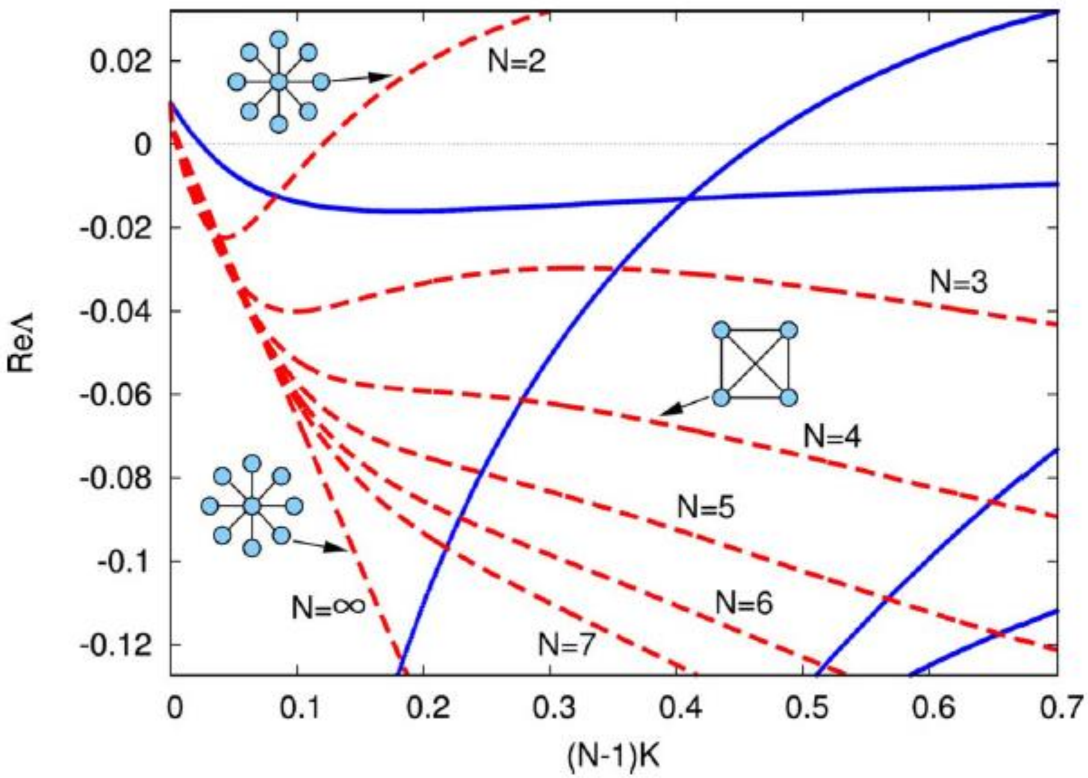
$$\dot{z}_j = f(z_j) + \sigma \sum_{n=1}^{N} a_{jn}[z_n(t-\tau) - z_j(t)],$$

$$\dot{\xi} = \mathbf{I}_N \otimes (\mathbf{J}_{0,m}^{\mp} - K\boldsymbol{\Psi}_m)\xi + K(\mathbf{A} \otimes \mathbf{R}_{n,m})\xi(t-\tau),$$

$$\dot{\zeta}_k(t) = \mathbf{J}_0^{\mp}\zeta_k(t) - K\mathbf{R}[\mu\zeta_k(t) - \nu_k\zeta_k(t-\tau)],$$

Networks of phase-reduced Stuart-Landau oscillators

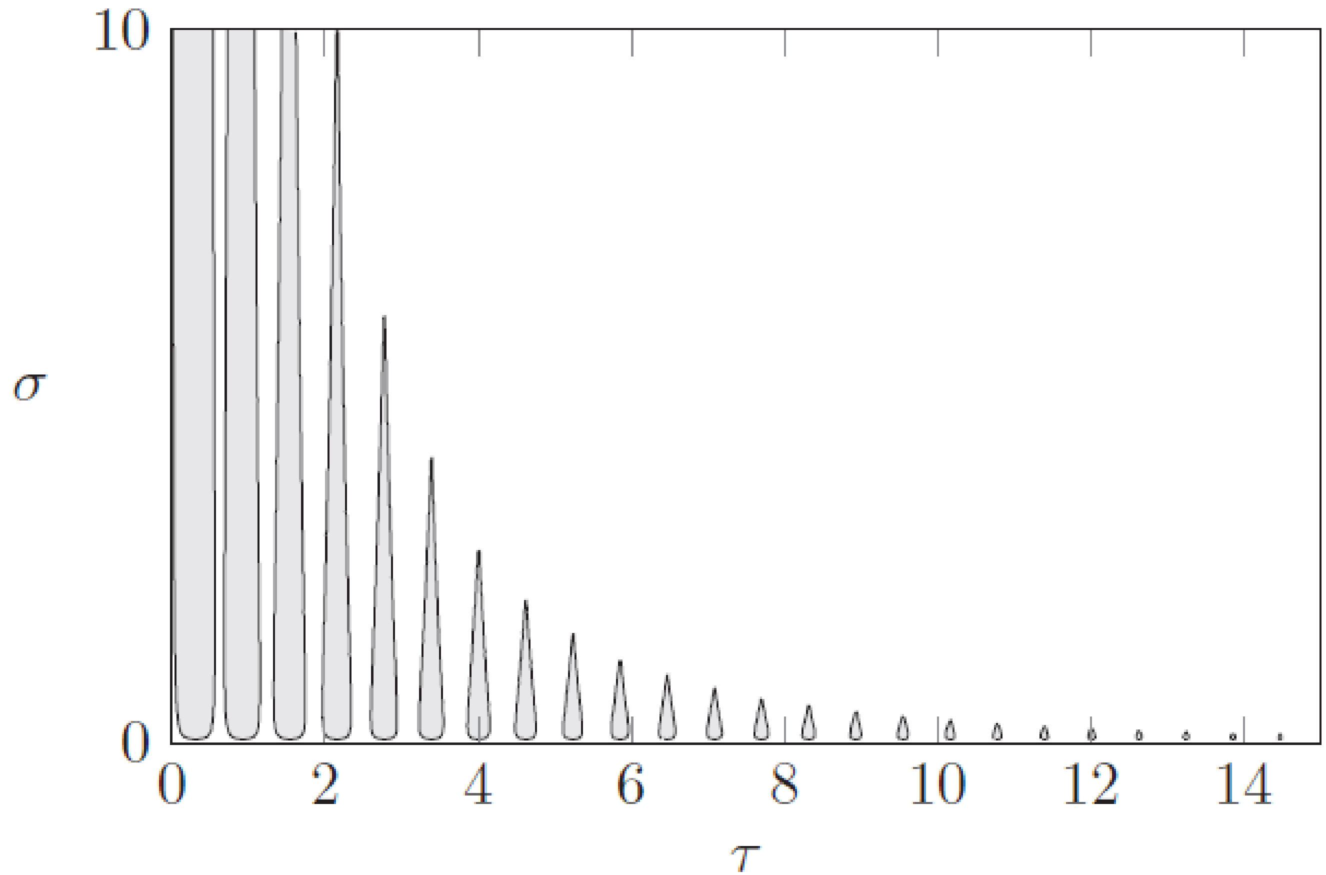
Choe, Dahms, Hövel, Schöll
PRE 2010



Extensions of the MSF approach

Homogeneous delay

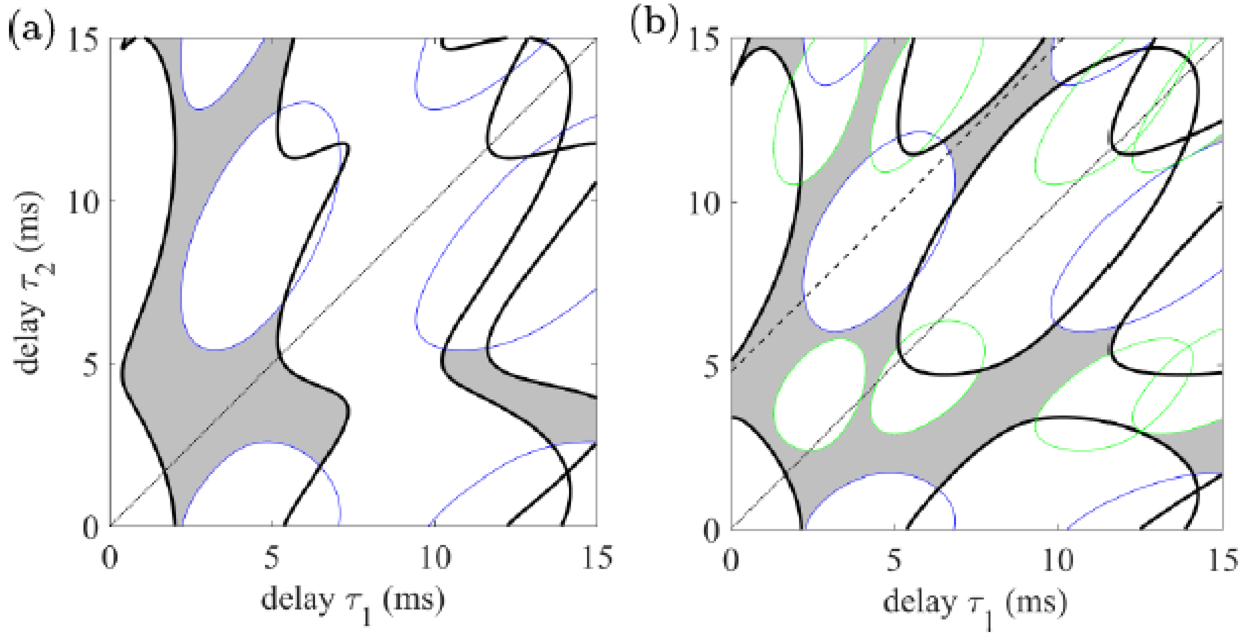
Amplitude death islands in a network of Lorenz chaotic oscillators



Huddy & Sun PRE 2016

Extensions of the MSF approach

Heterogeneous delay



Stability charts for two types of networks of HH neurons with two delays

Otto, Radons, Bachrathy, Orosz PRE 2018

Extensions of the MSF approach

Heterogeneous delay

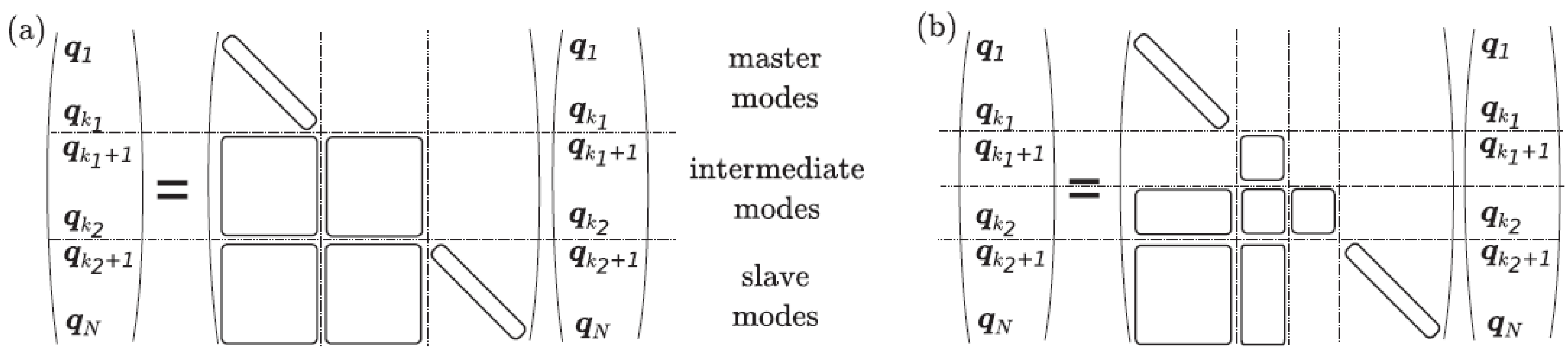


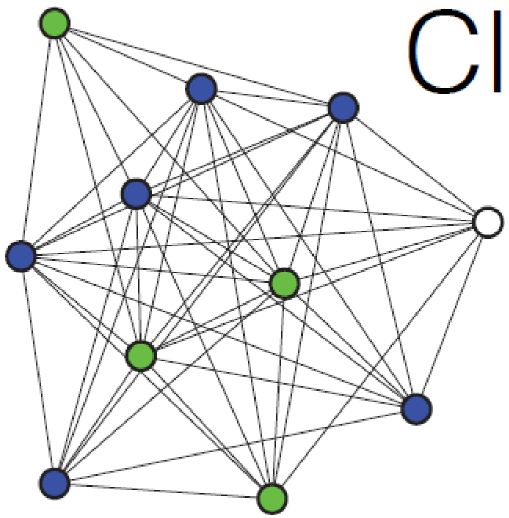
FIG. 2. (a) Structure of Eq. (21) with separation into master modes ($\mathcal{U}_k$ contains only numbers), slave modes ($\mathcal{V}_k$ contains only numbers), and intermediate modes. Only the squares and the diagonal stripes are nonempty. The stripes at the main diagonal are associated with Eq. (23) and determine the stability of the master and the slave modes, respectively. The intermediate modes are driven by the master modes and both can drive the slave modes. (b) Structure of Eq. (21) after additional decomposition of the intermediate modes. The two small squares at the main diagonal of the intermediate modes are associated with two blocks similar to Eq. (25), specifying the stability of the intermediate modes.

Eigenmode decomposition

Otto, Radons, Bachrathy, Orosz PRE 2018

Extensions of the MSF approach

Clusters (and Computational Group Theory)

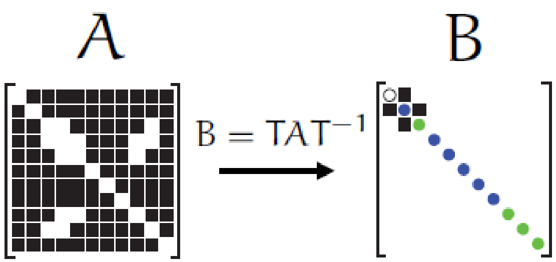


5,760 symmetries
3 clusters

GAP - Groups, Algorithms, Programming: a System for Computational Discrete Algebra
http://www.gap-system.org

$$\dot{z}_i = F(z_i) + \sigma \sum_j A_{ij} H(z_j)$$

Irreducible representations of the graph automorphism



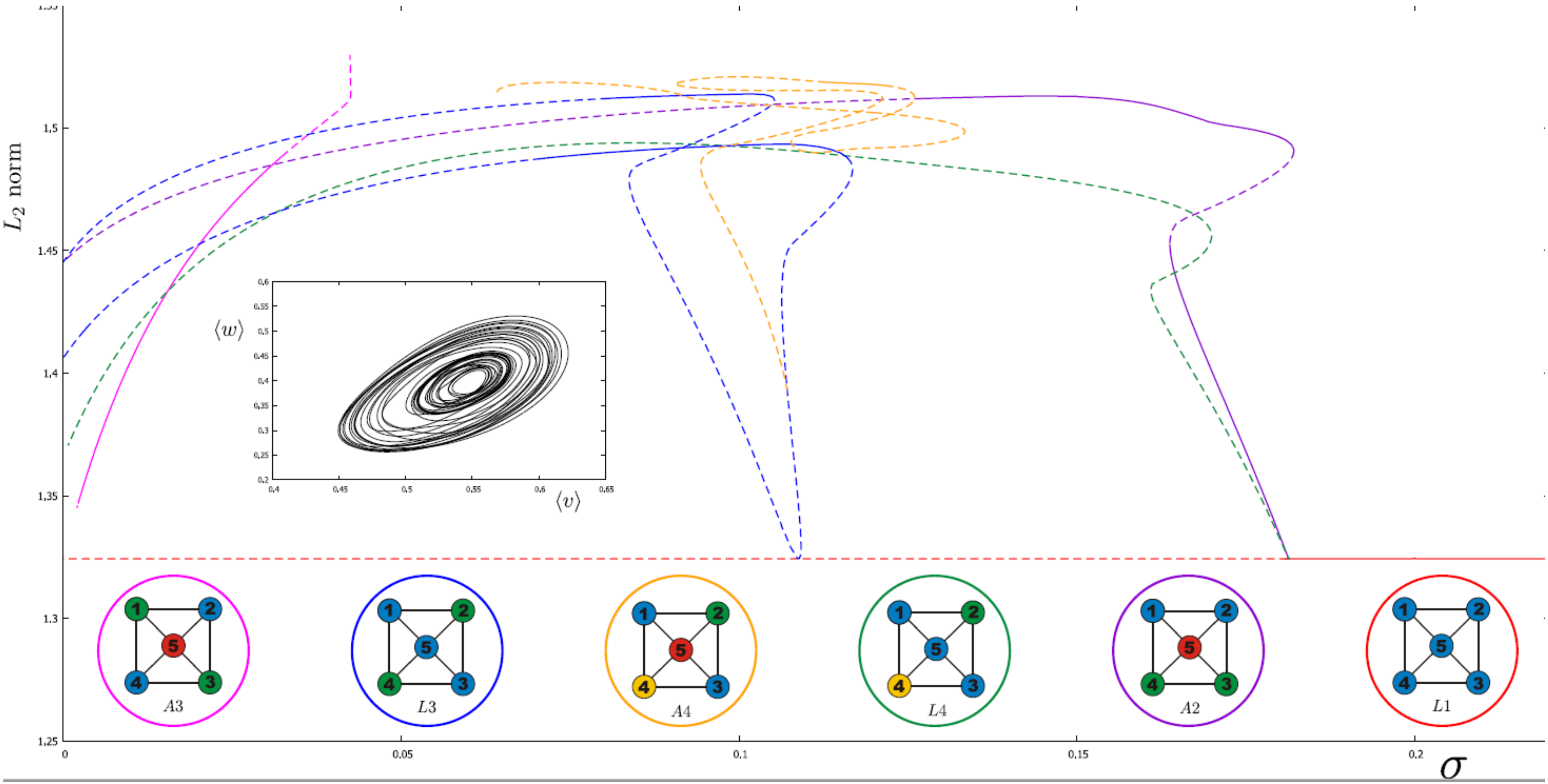
Nice variational formulation for M clusters

$$\dot{y} = \left[\sum_{m=1}^{M} E^{(m)} \otimes DF(s_m) + \sigma B \otimes I_n \sum_{m=1}^{M} J^{(m)} \otimes DH(s_m) \right] y$$

L M Pecora, *et al*. Cluster synchronization and isolated desynchronization in complex networks with symmetries. Nature Communications, 5(4079), 2014.

Extensions of the MSF approach

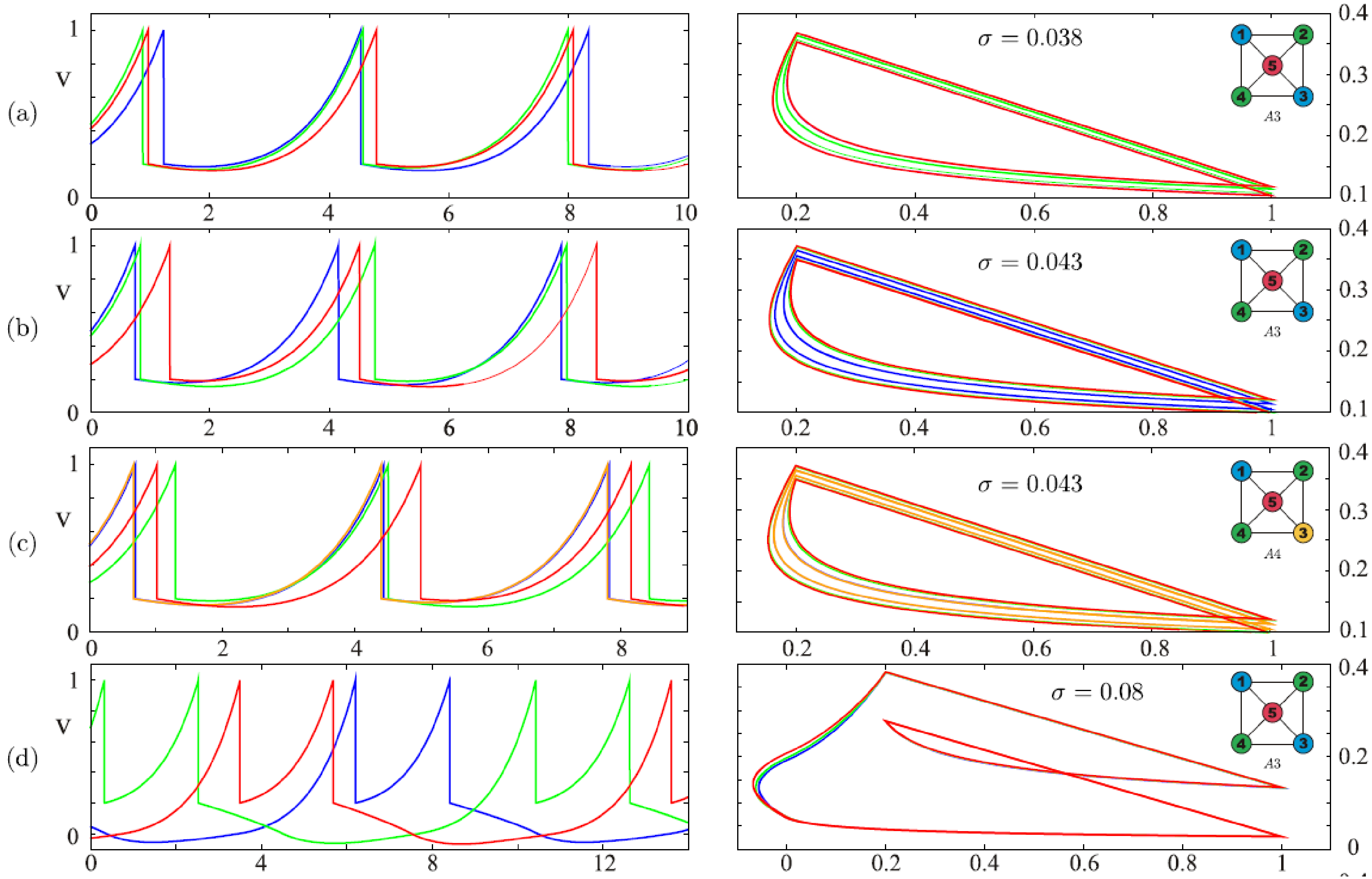
Application to a network of piecewise Morris-Lecar neurons



Nicks, Chambon & Coombes, PRE 2018

Extensions of the MSF approach

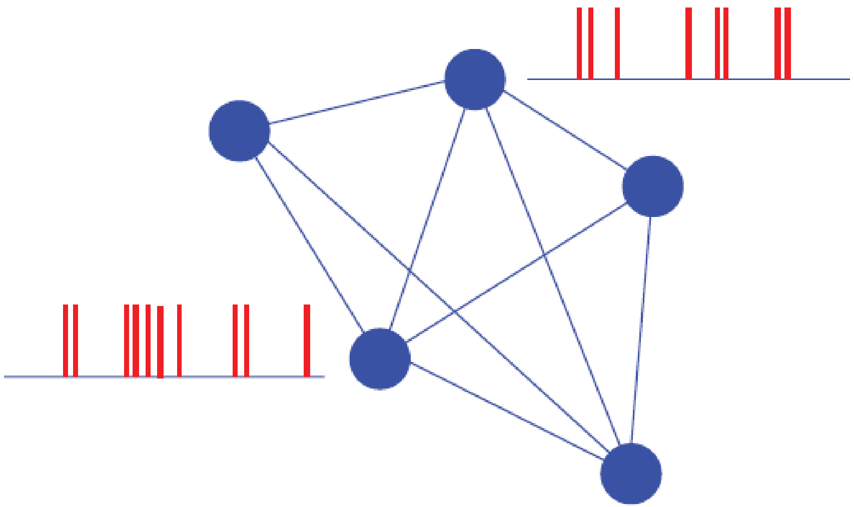
Application to a network of planar IF neurons

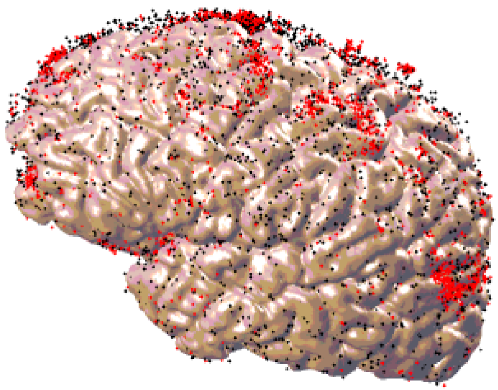


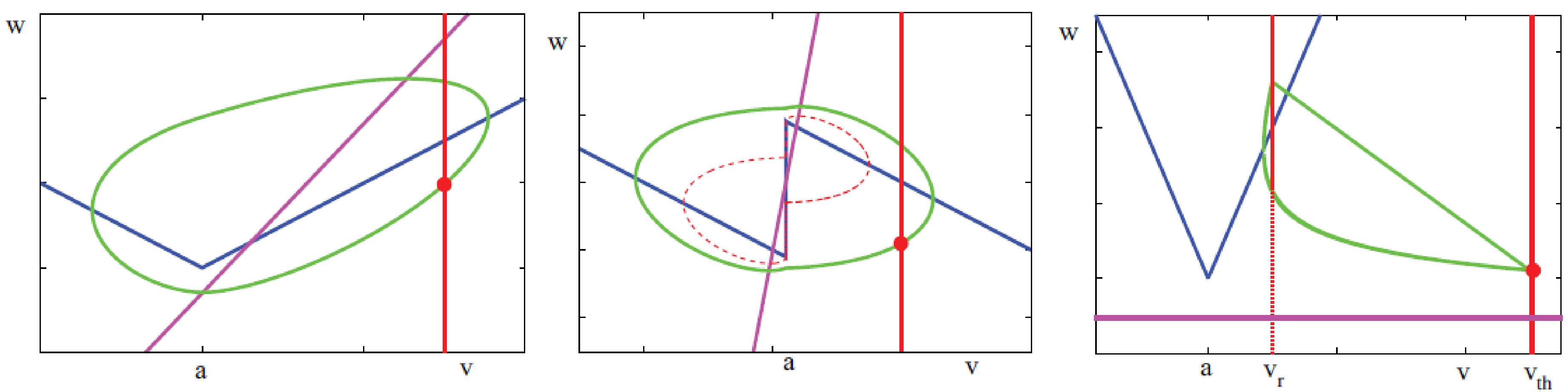
Nicks, Chambon & Coombes, PRE 2018

Extensions of the MSF approach

… strong coupling, event driven interactions, …



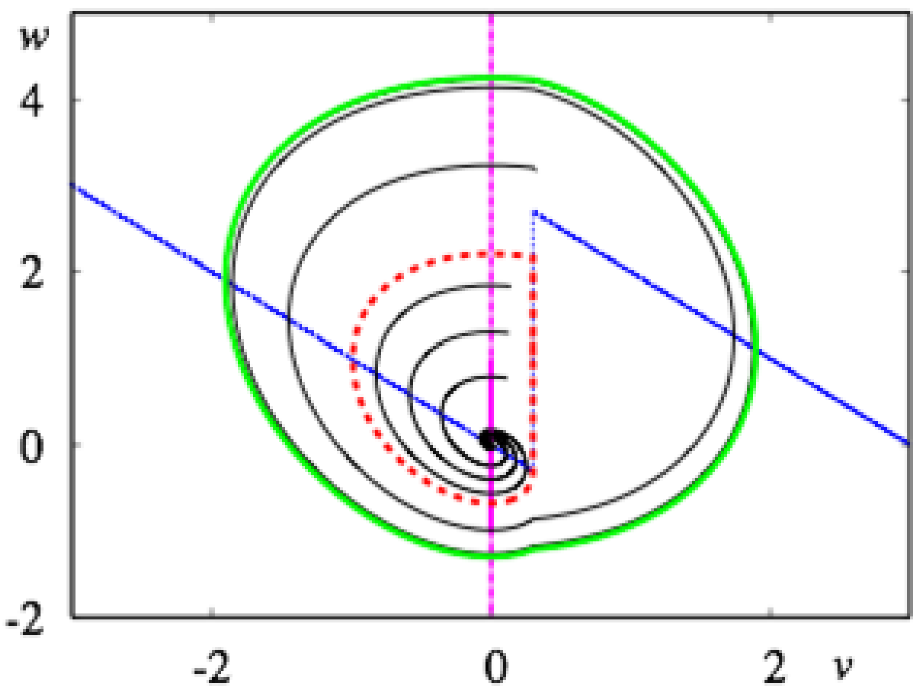




Challenge of studying networks of non smooth and discontinuous ***threshold*** elements.

Extensions of the MSF approach

- Nonsmooth systems – Coombes & Thul EJAM 2016
- Intrinsically discontinuous systems – e.g. integrate-and-fire
- PWL caricatures of a wide range of nonlinear models
  - Quantities can be written down explicitly
  - Much more mathematical analysis can be done



McKean model
Tonnelier, SIAM J. Appl. Math., 2002.

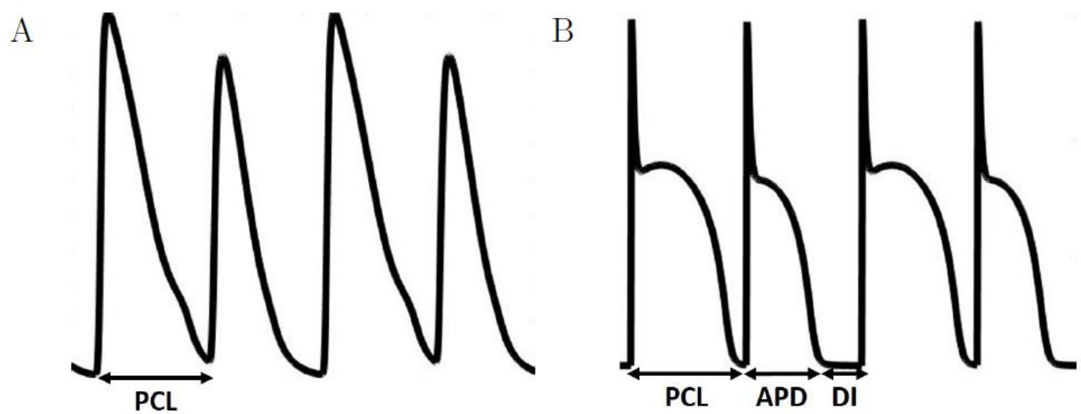
Summary – MSF for piecewise linear systems

	Continuous trajectories/ vector fields	Discontinuous trajectories / vector fields
Continuous interactions	A Matrix exponentials	B Matrix exponentials with saltation matrices
Discontinuous interactions	C Glass networks	D Ordering problem

Application – cardiac alternans

A

- One precursor to arrythmias and heart attacks are alternans
- Characterized by alternating beats of different length or contraction strength
- (Contraction strength corresponds to amplitude of calcium oscillations)
- Bidirectional coupling between Calcium cycling and voltage has led to an ongoing debate about which induces which



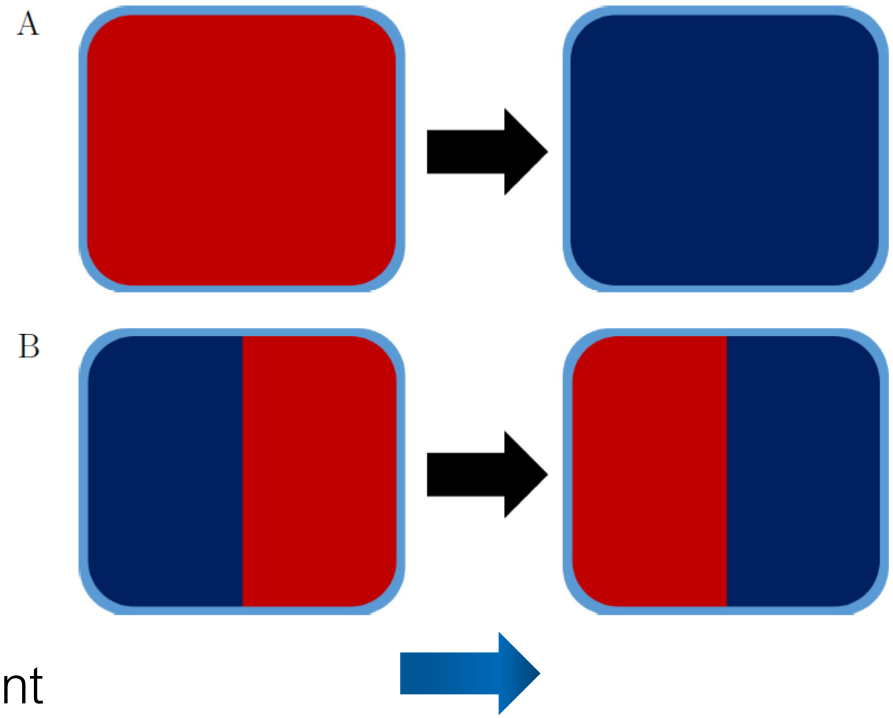
Calcium alternans

Voltage alternans

Subcellular alternans

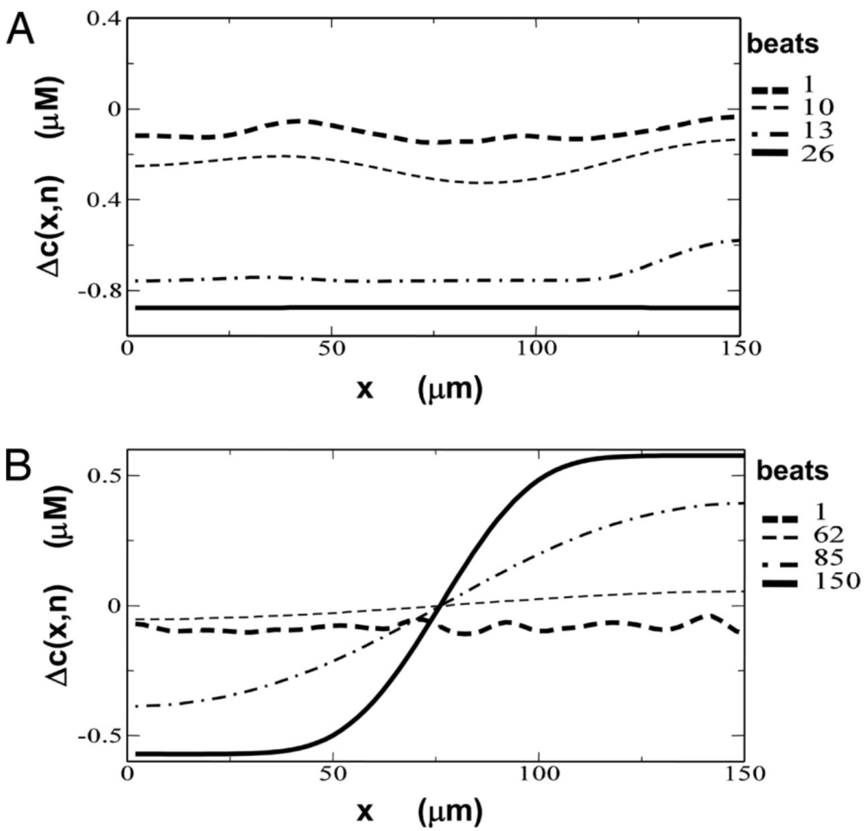
Alternans also occur at the subcellular level

Spatially concordant



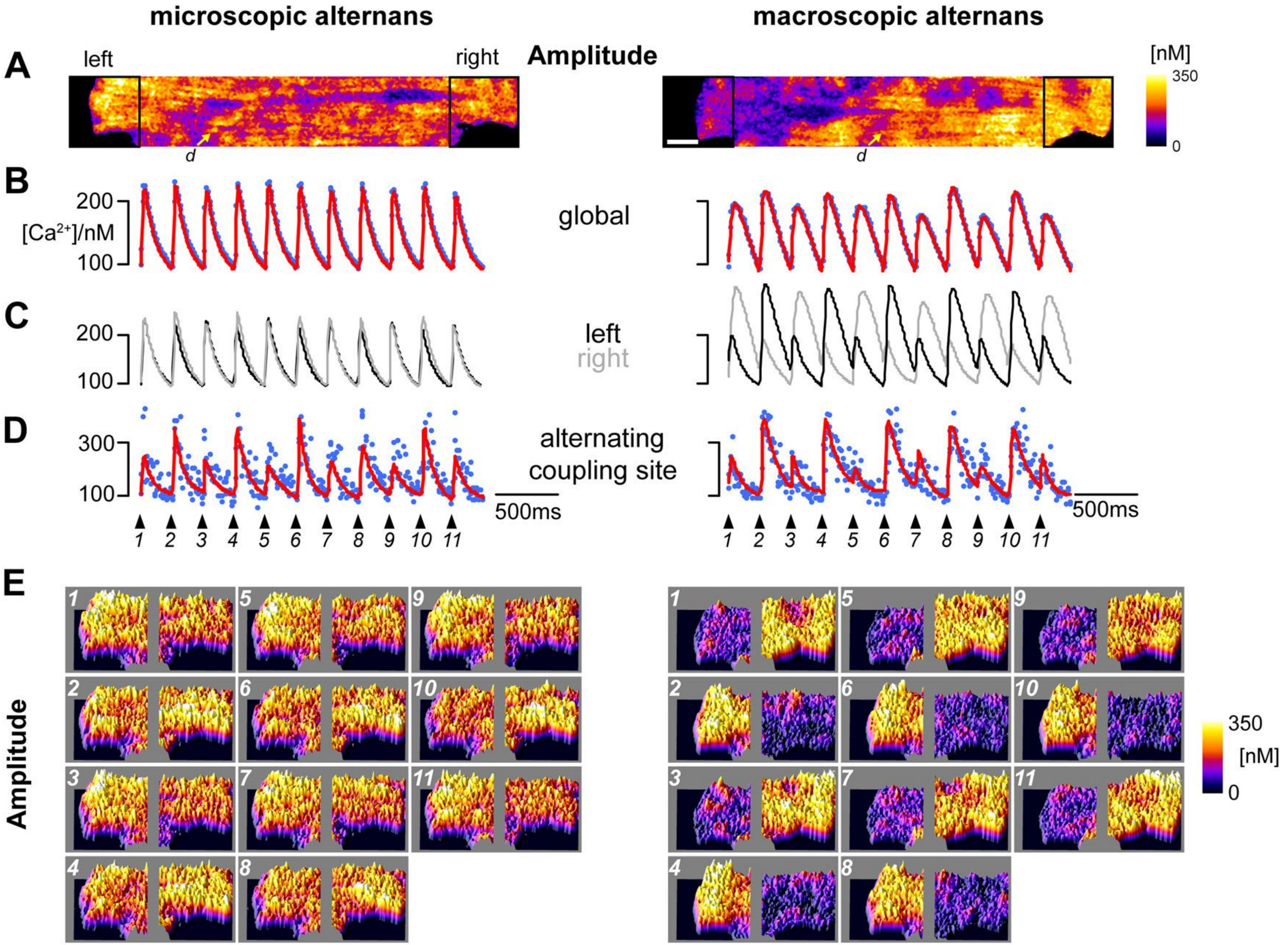
Spatially discordant

Consecutive heartbeats



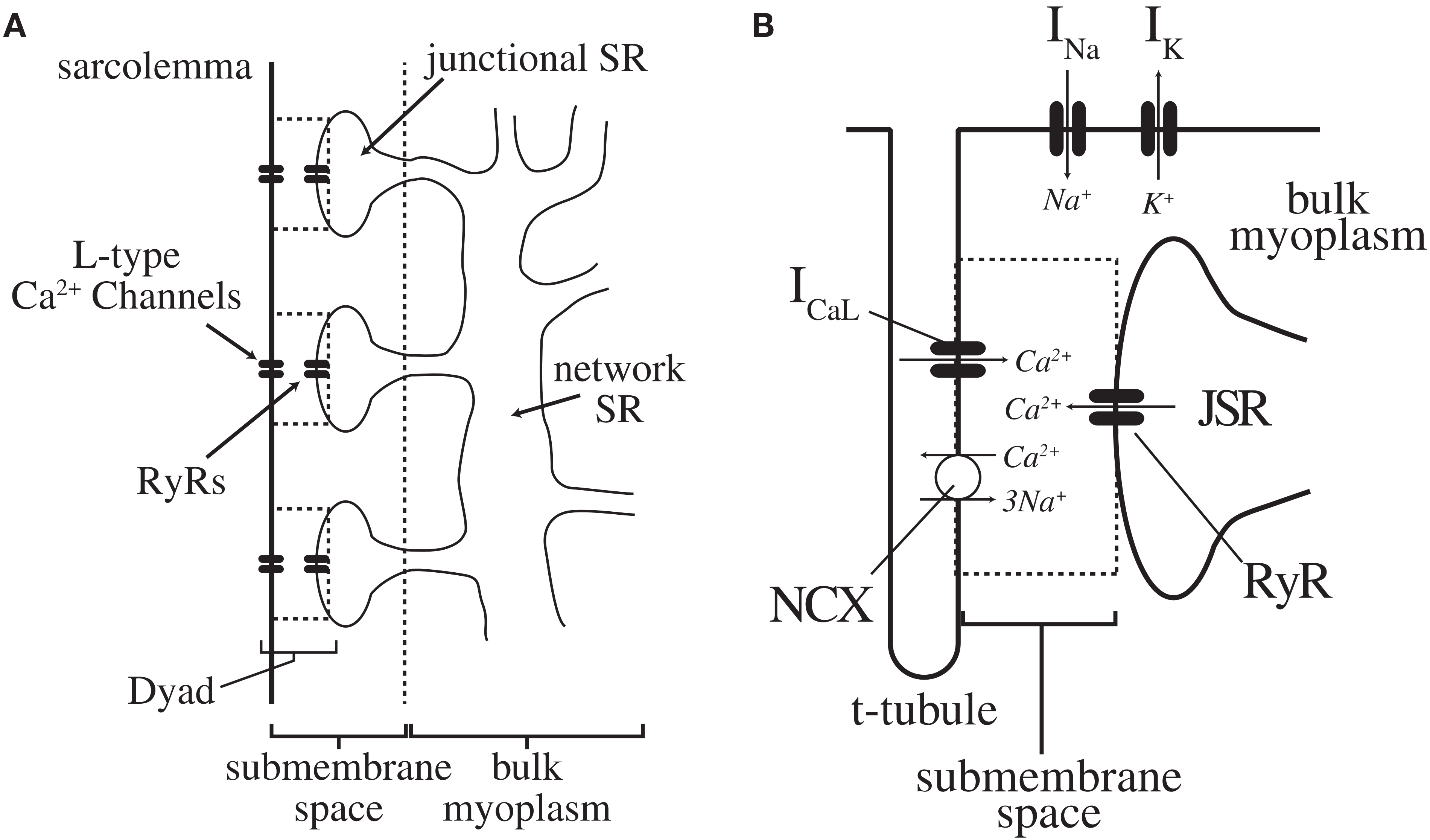
Shiferaw & Karma, PNAS 2016

Subcellular alternans



Tian, Kaestner, Lipp, Circ Res 2012

Calcium cycling in a heart cell



Gaeta & Christini, Front Physiol 2012

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Mathematical modelling – Shiferaw-Karma model

5 ODEs for 1 CRU

$$\frac{dc_s}{dt} = \beta(c_s)\left[\frac{v_i}{v_s}\left(I_r - \frac{c_s - c_i}{\tau_s} - I_{CaL}\right) + I_{NaCa}\right],$$

$$\frac{dc_i}{dt} = \beta(c_i)\left[\frac{c_s - c_i}{\tau_s} - I_{up}\right],$$

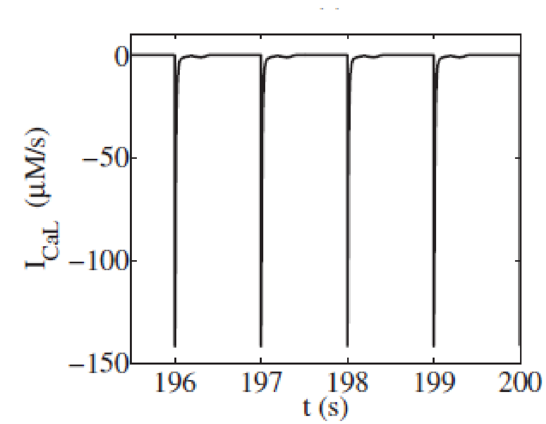
$$\frac{dc_j}{dt} = -I_r + I_{up},$$

$$\frac{dc_j'}{dt} = \frac{c_j - c_j'}{\tau_a},$$

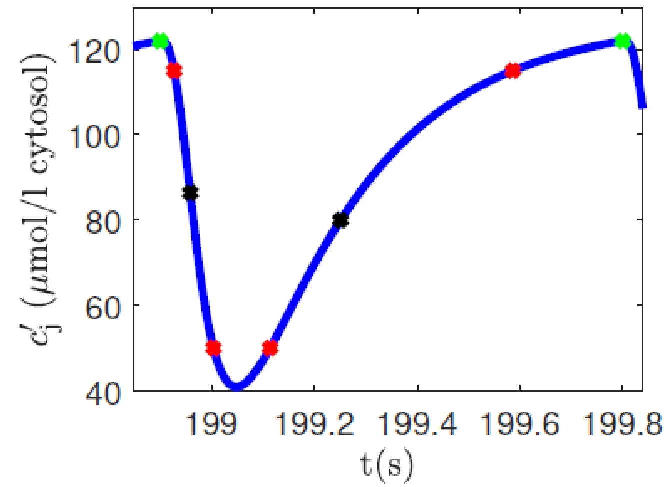
$$\frac{dI_r}{dt} = -gI_{CaL}Q(c_j') - \frac{I_r}{\tau_r}.$$

Voltage-dependent currents (non-autonomous)
When voltage is clamped, these become time-dependent switches

State-dependent switch

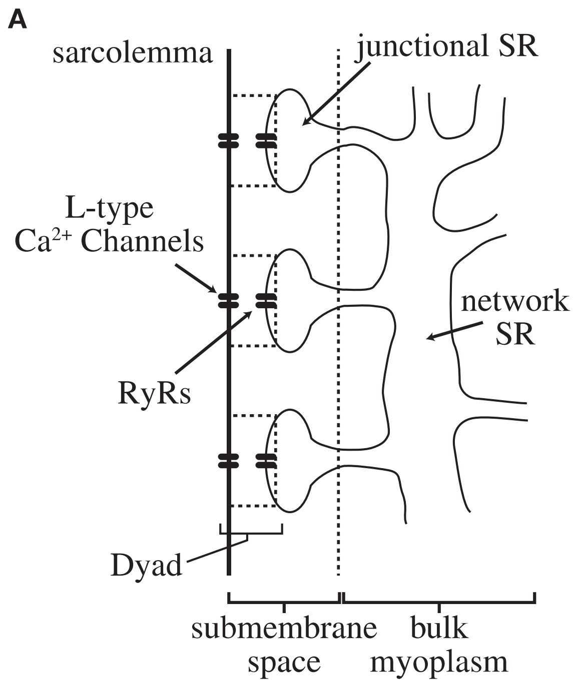


Stiff problem

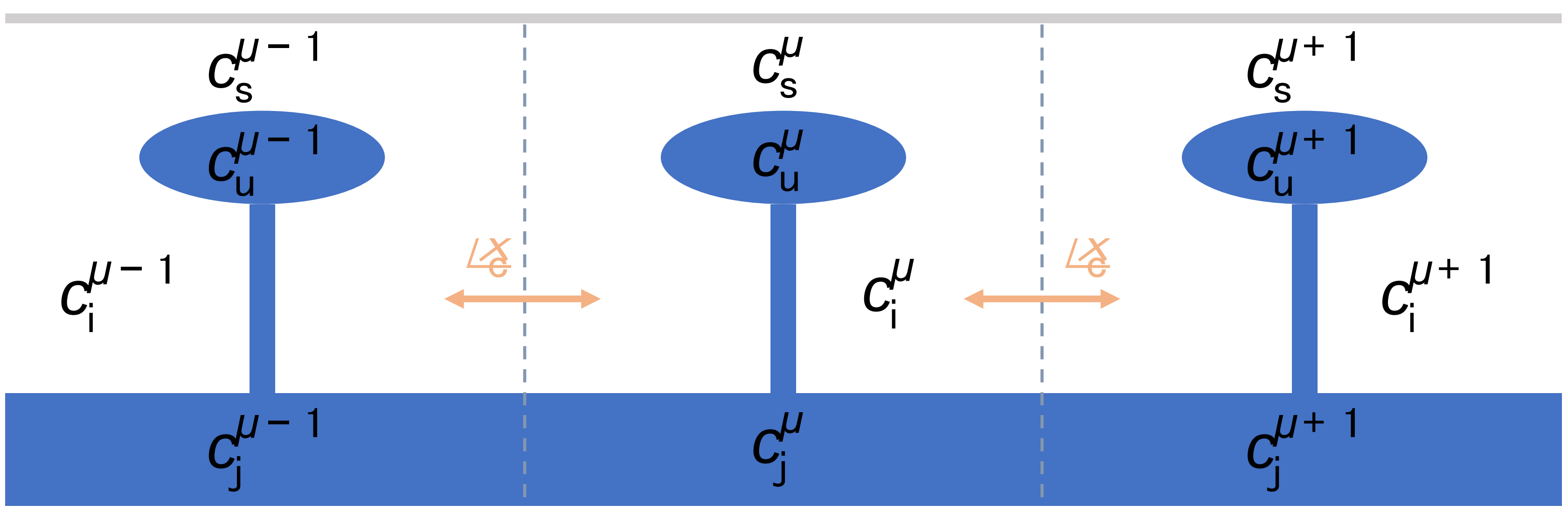


Thul & Coombes, Chaos 2010

CRU network

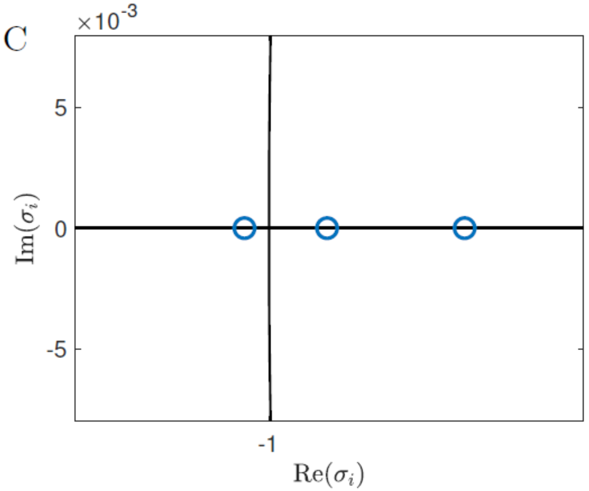


Diffusive coupling
(in different compartments)

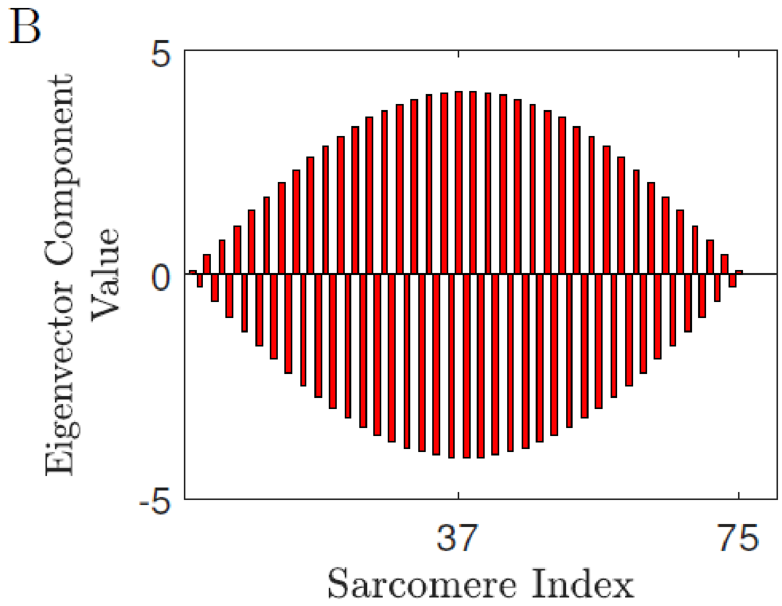


From stability to patterns

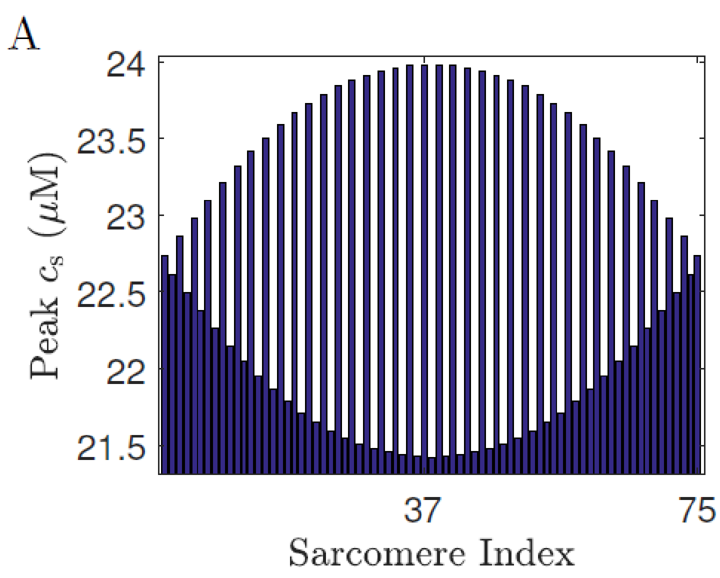
A 1-D network of 75 CRUs



One eigenvalue of the stability problem just goes unstable



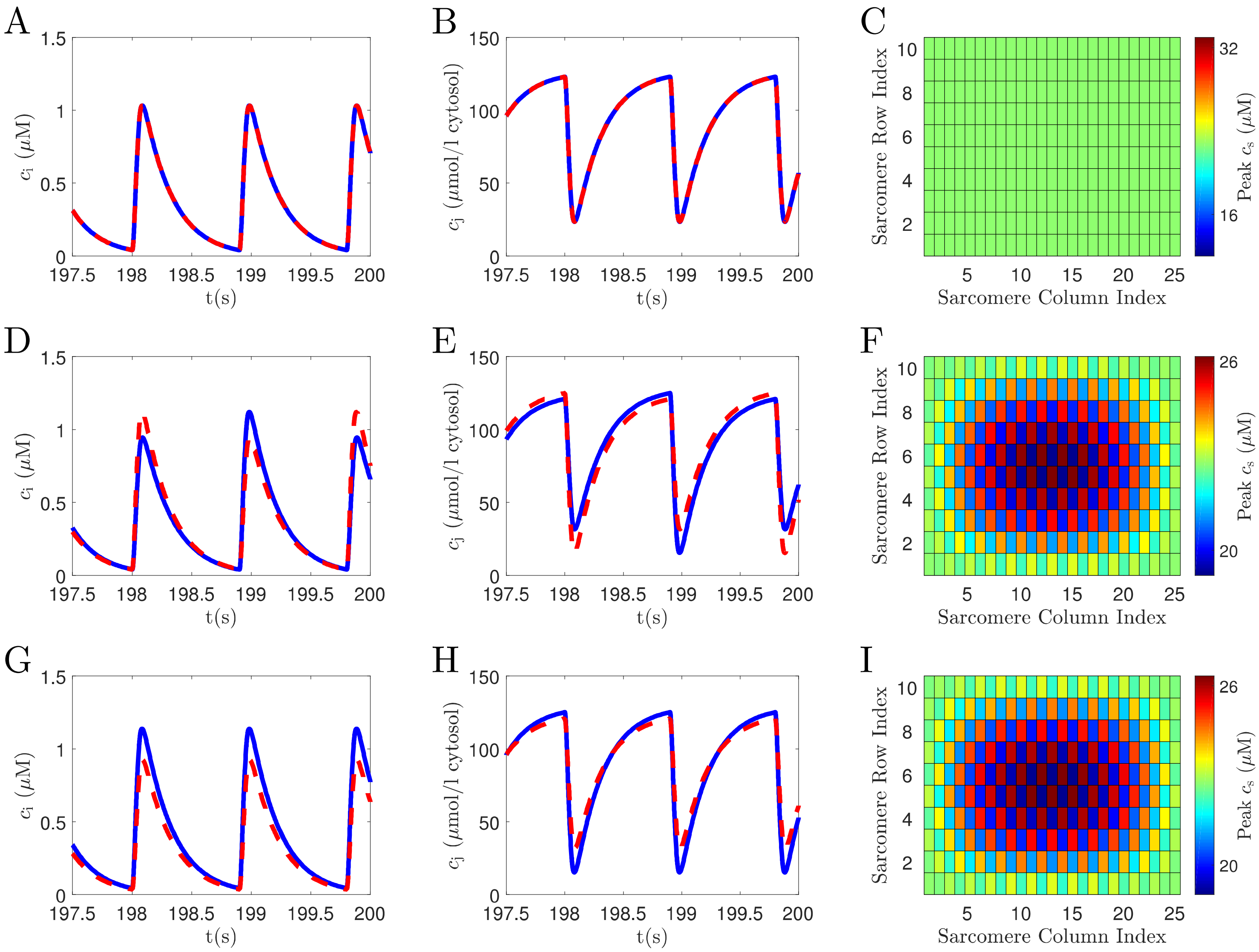
Corresponding network eigenvector shows the alternan pattern



Simulation of the full network

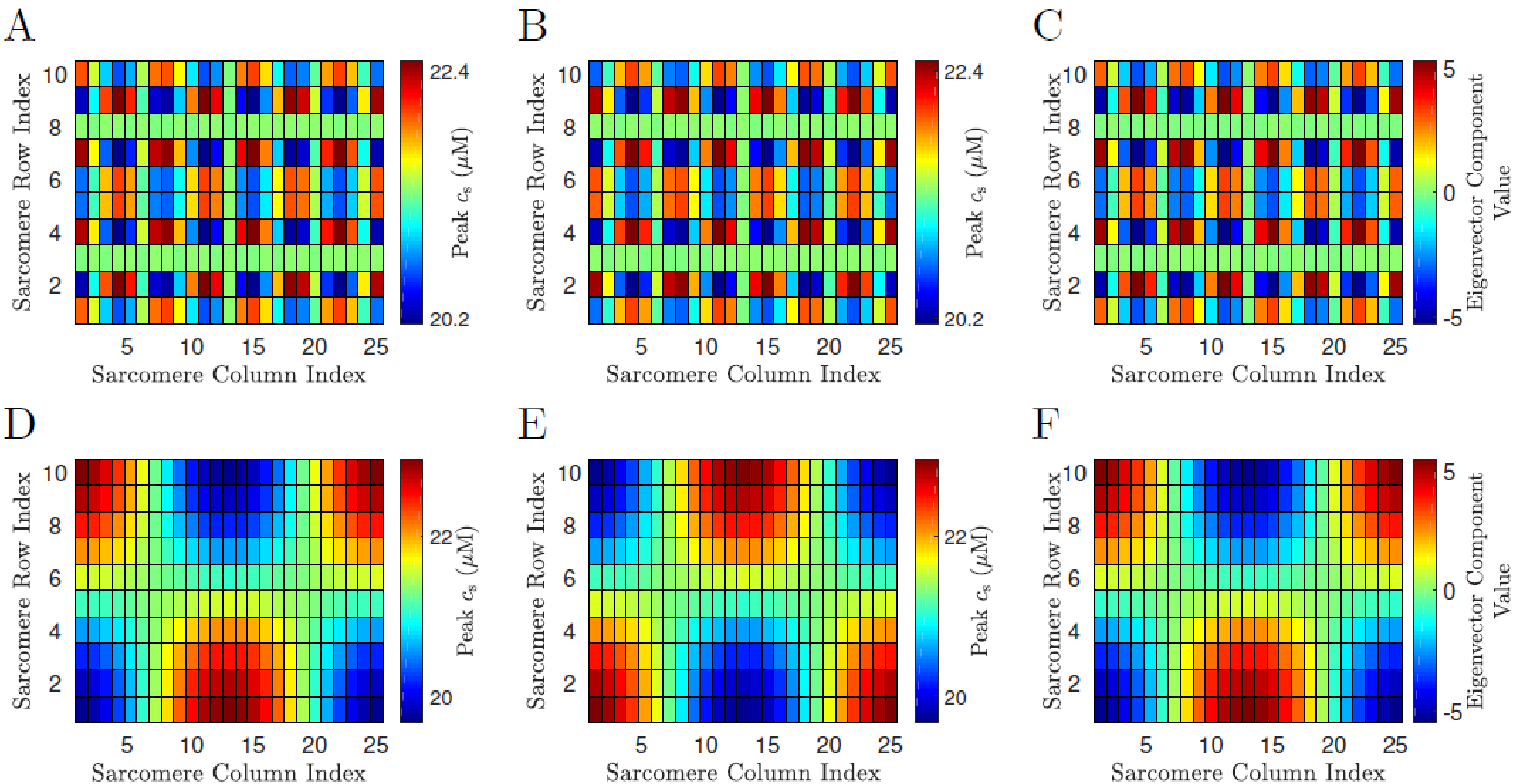
Veasy, Lai, Coombes, Thul, J Theor Biol 2019

A 2-D network of 25x10 CRUs



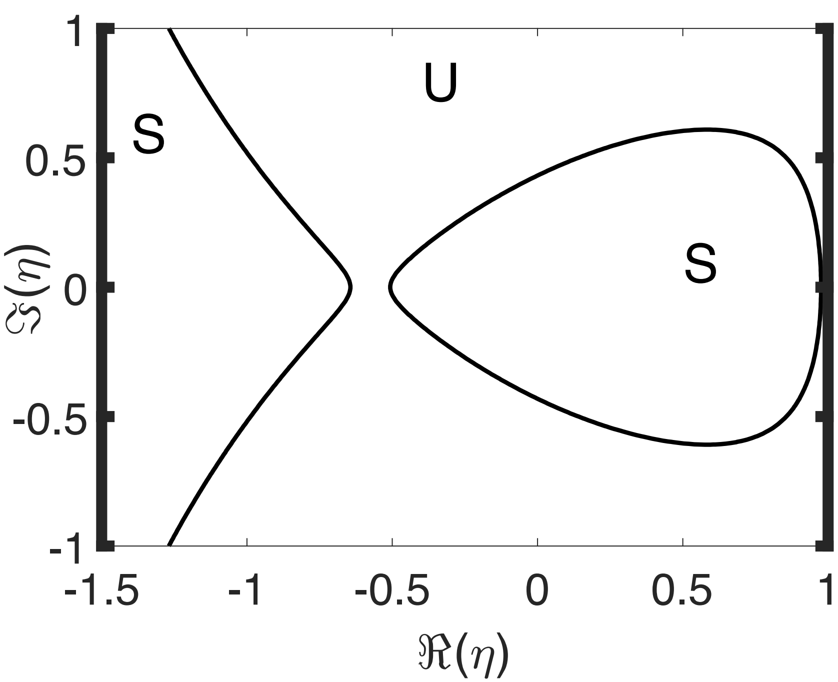
Eigenvalues predict patterns

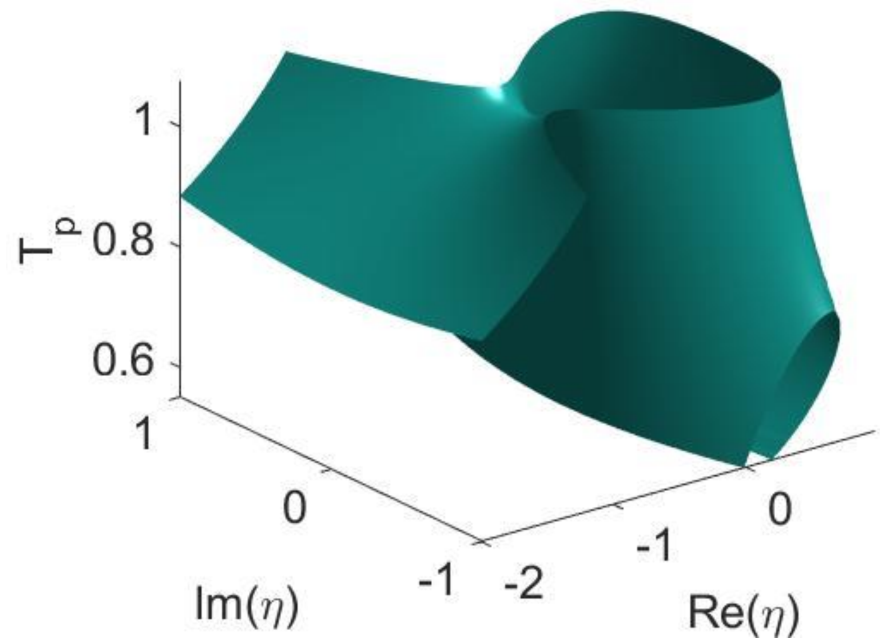
A 2-D network of 25x10 CRUs



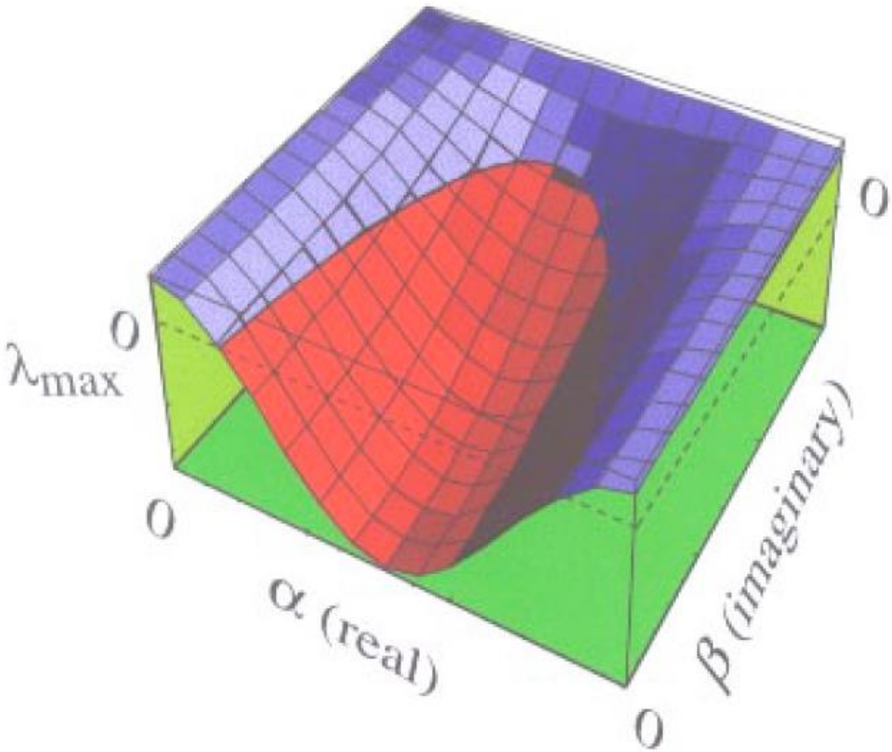
MSF analysis

- Using a PWL version of the model allows us to simplify MSF analysis
- Nonlinear equations for Floquet exponents can now be replaced by simple matrix exponentials
- We can then combine the MSF with other parameters to systematically carry out parameter sweeps
- This is done semi-analytically, so is computationally cheap





Recap



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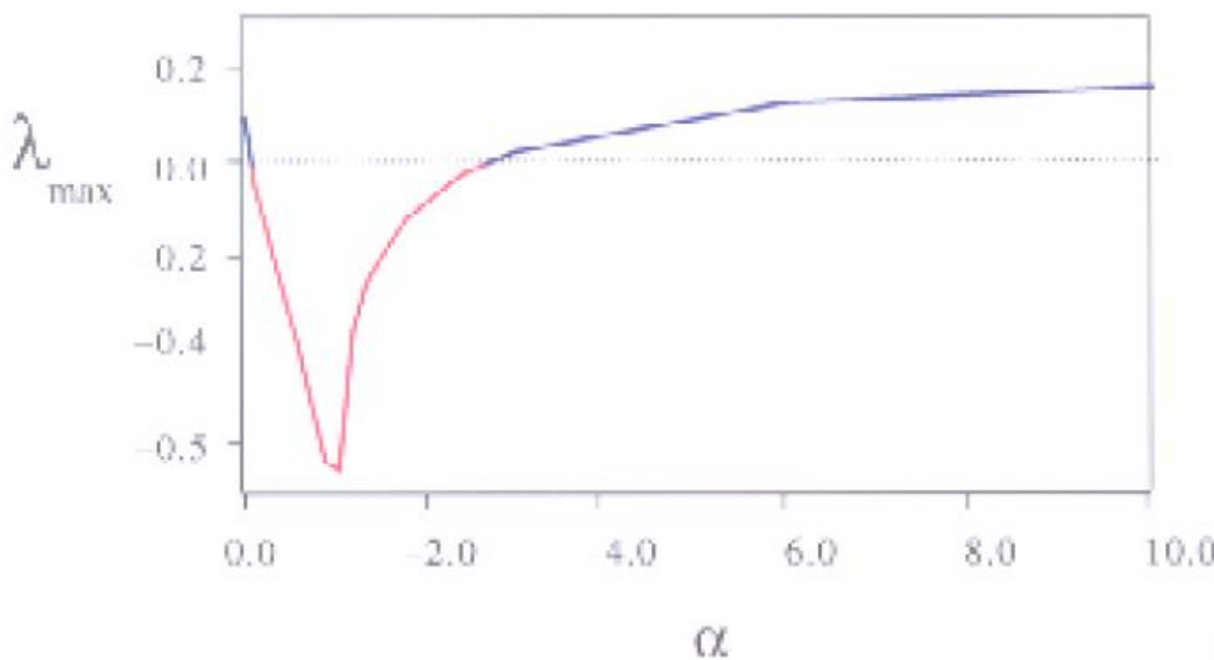

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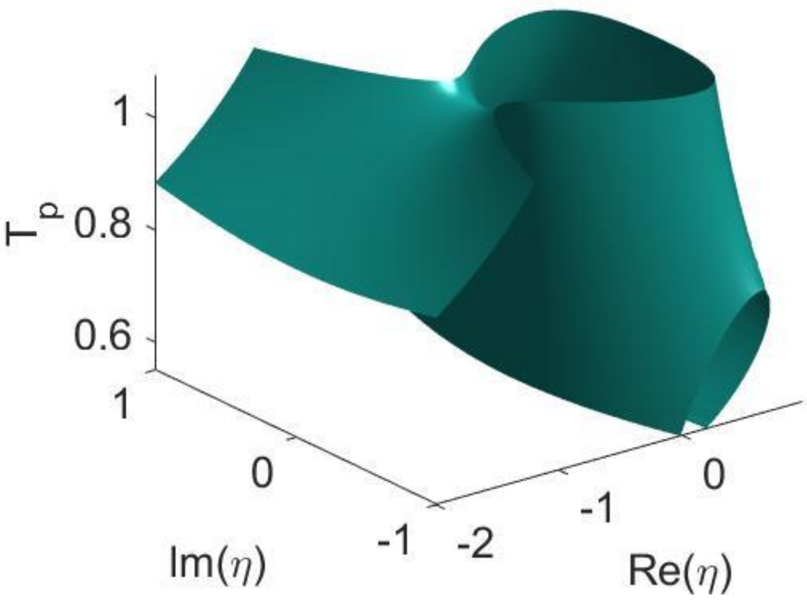

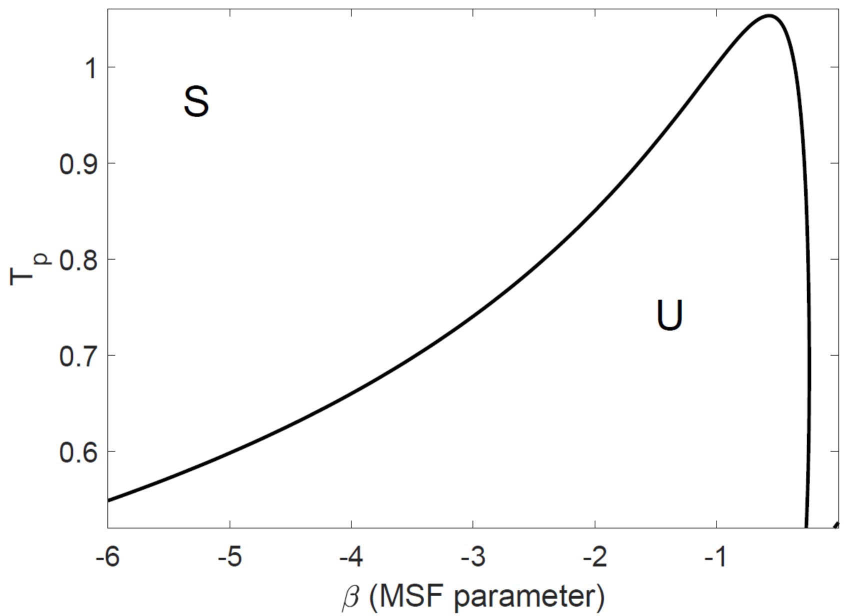

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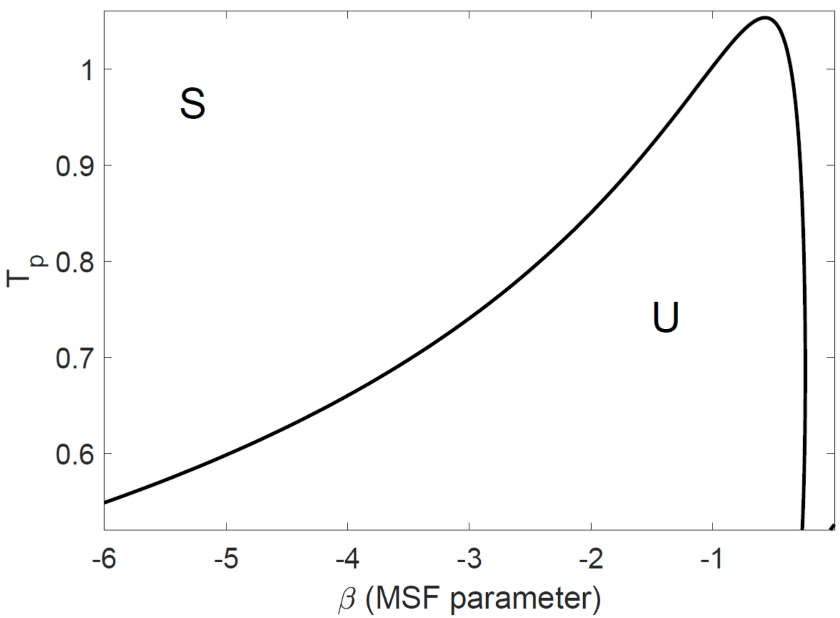
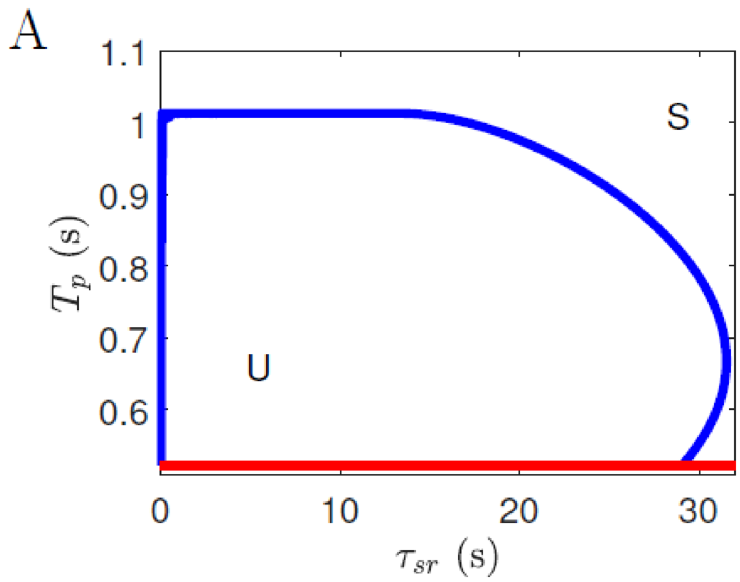


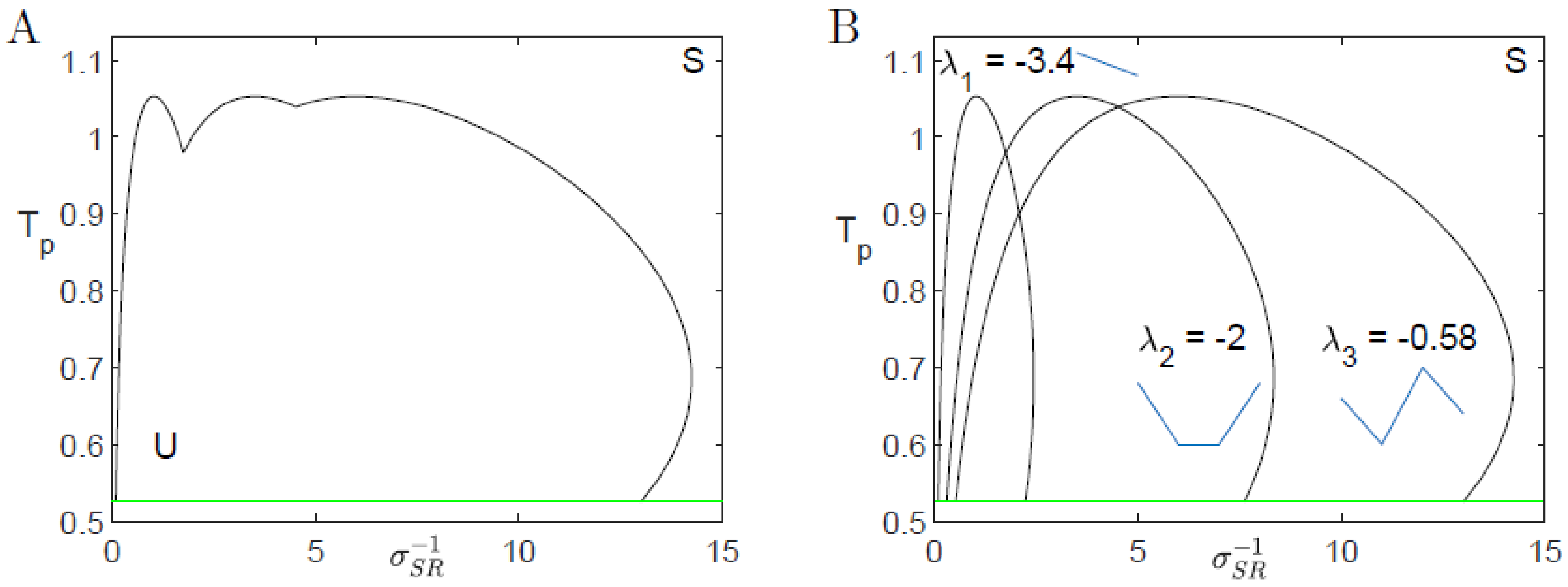


For a given network:

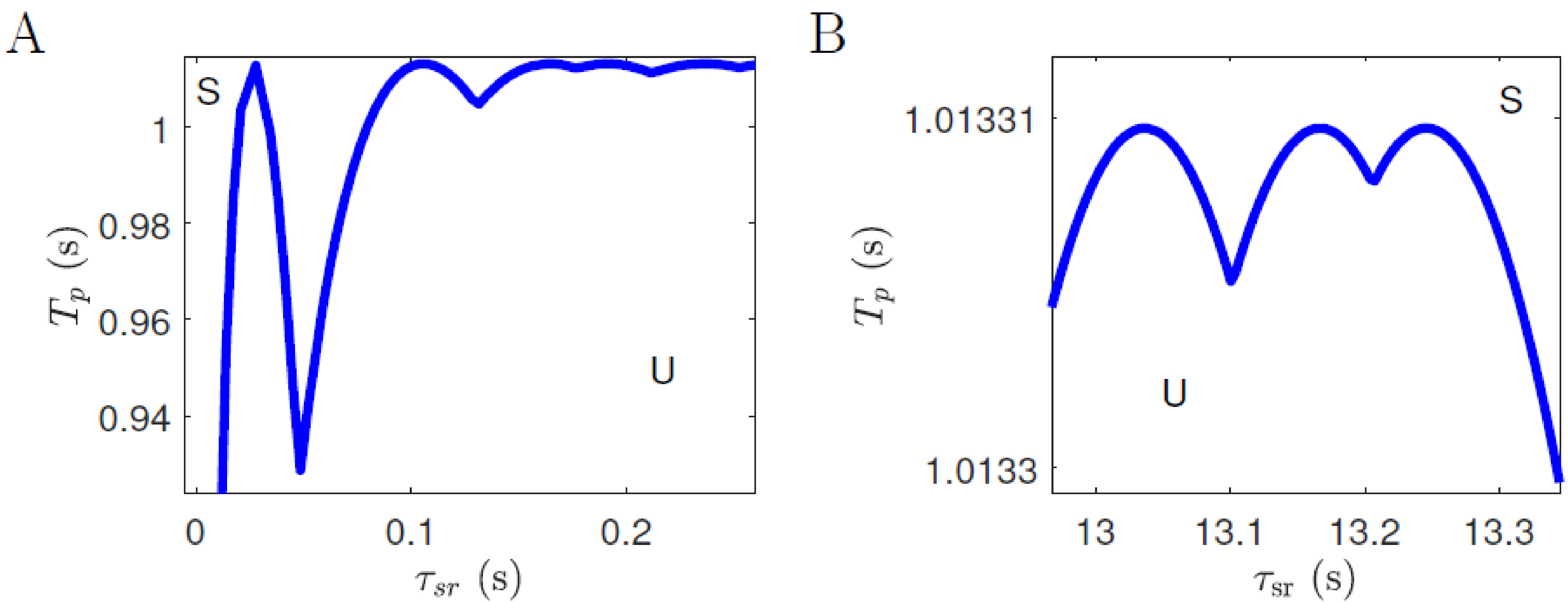
Each eigenvalue gives an interval of instability, and the union of these intervals for all the network eigenvalues gives a traditional bifurcation diagram

MSF analysis

Example for a toy network (N=4)

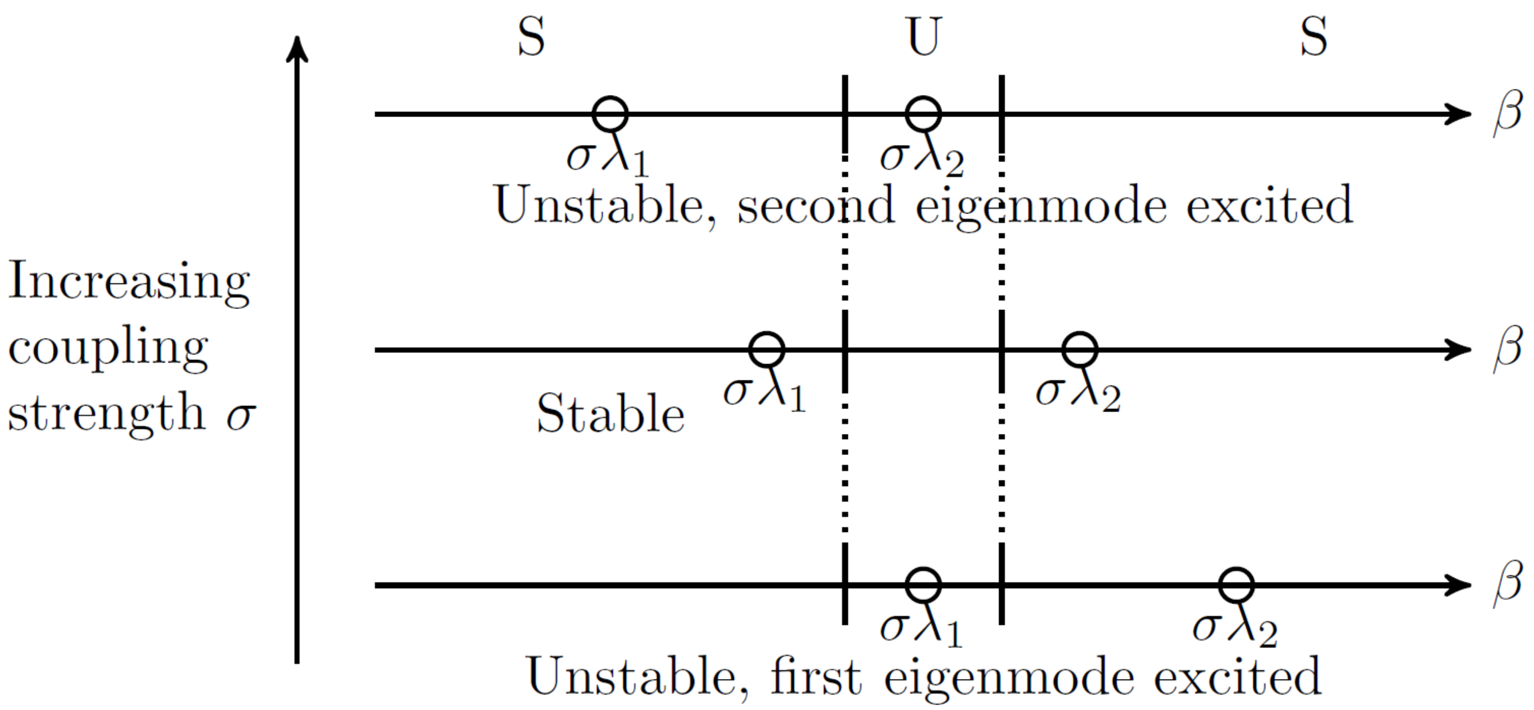


Finding exotic patterns



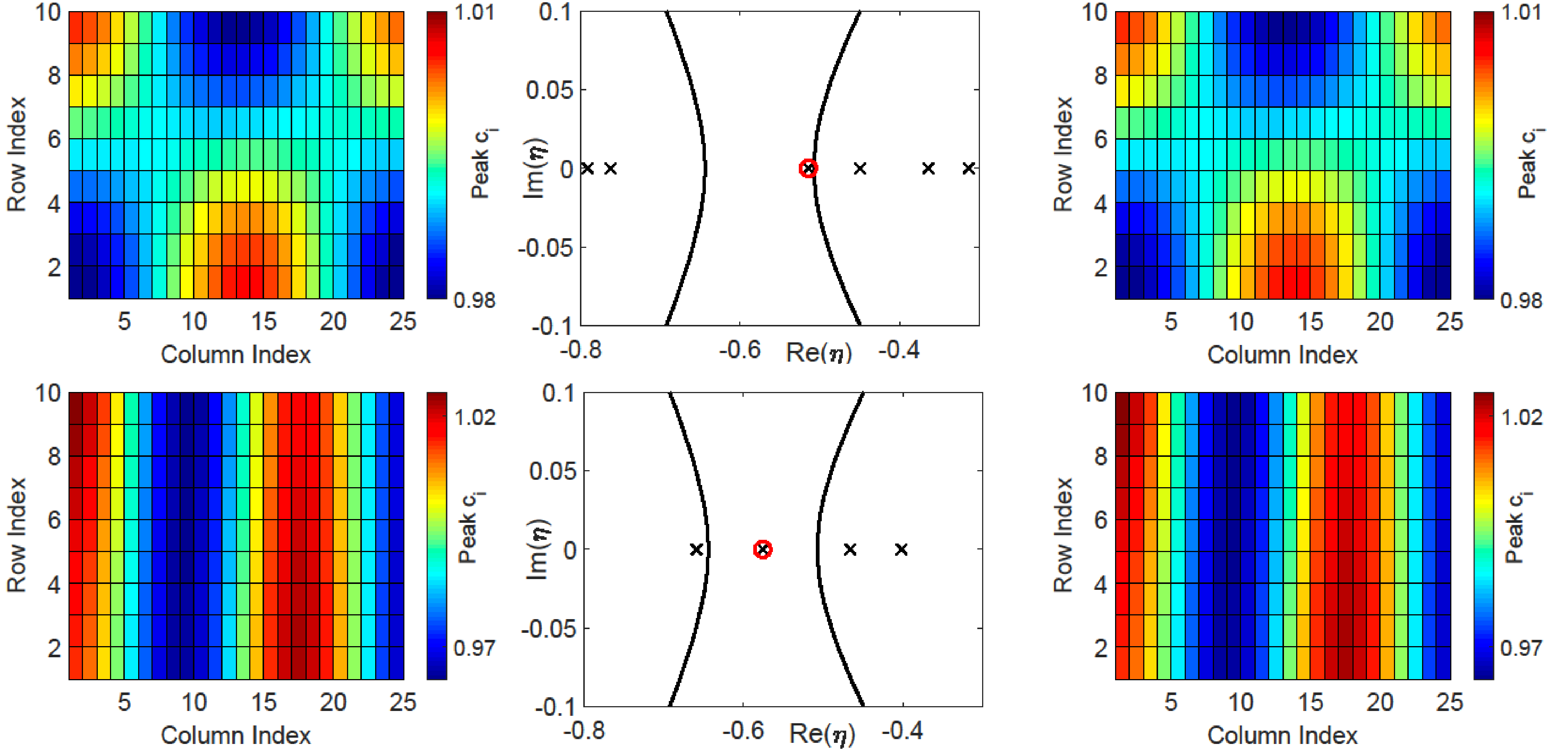
These explain mysterious “bumps” in the bifurcation diagram that we found in a computational parameter sweep

Finding exotic patterns



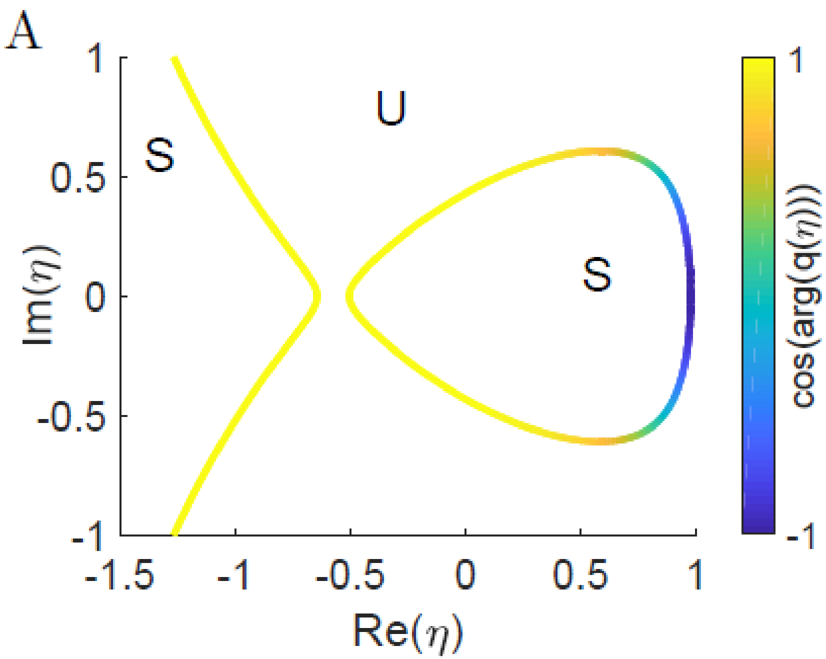
In areas where only a small interval is unstable, we can see that small parameter variation can cause drastic changes in stability or different patterns being exhibited.

Finding exotic patterns

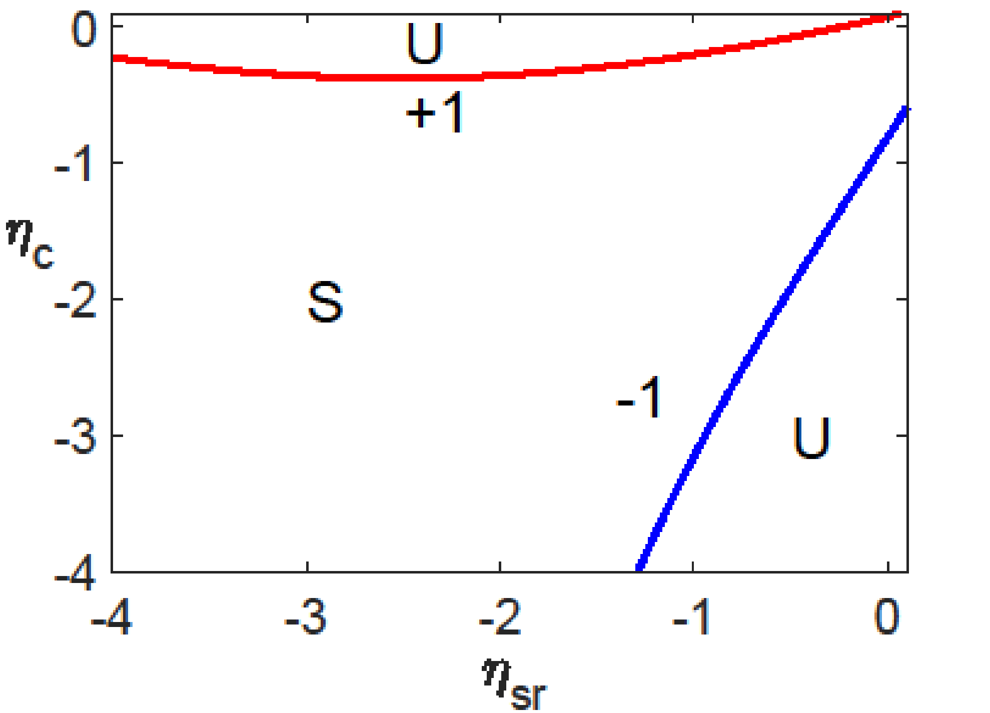


Lai, Veasy, Coombes, Thul, under review

Different types of bifurcations

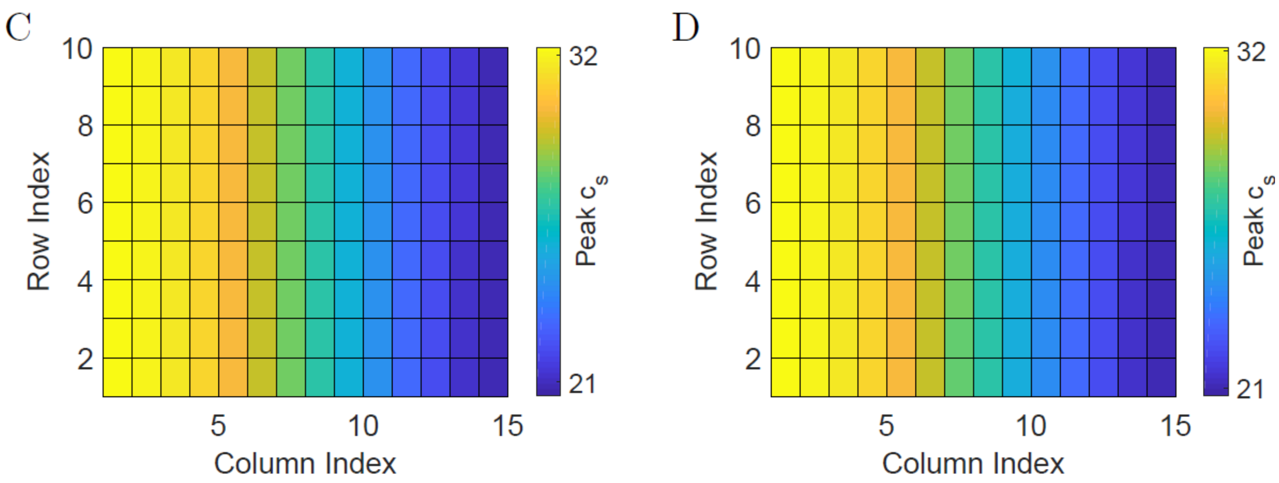


We can also add information like how the eigenvalue for the stability problem leaves the unit disc



Alternans are traditionally associated with period-doubling (-1) bifurcations, but we find a +1 bifurcation when coupling through the SR is high

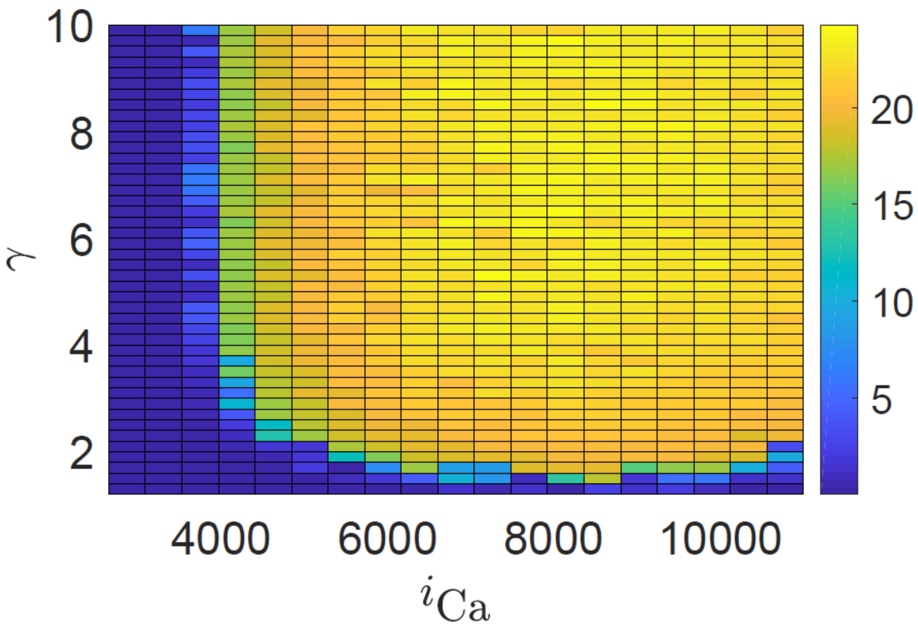
Using PWL analysis to guide computation



+1 bifurcation

We used this to look for this new bifurcation in the full computational model

By unravelling our assumptions one-by-one, we identify the key factors as the calcium inactivation gate in the L-type channel and buffering



Lai, Coombes, Thul, under review

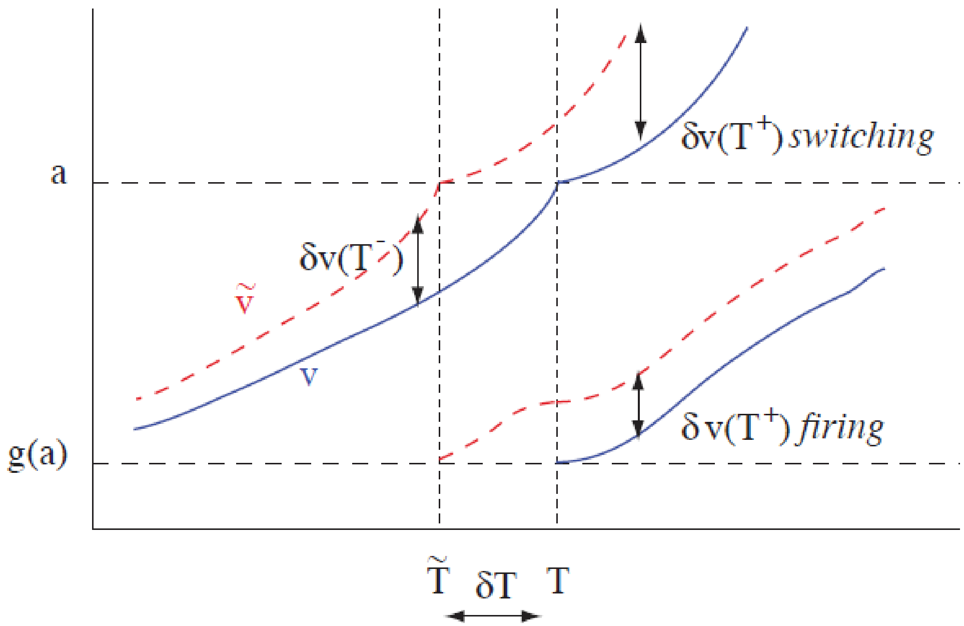
Summary – MSF for piecewise linear systems

	Continuous trajectories/ vector fields	Discontinuous trajectories / vector fields
Continuous interactions	A Matrix exponentials	B Matrix exponentials with saltation matrices
Discontinuous interactions	C Glass networks	D Ordering problem

Extensions of the MSF approach - discontinuities

B

- In the previous example, trajectories were continuous
- For discontinuous trajectories, this approach has to be complemented by saltation matrices
- Evolution of perturbations through a discontinuity



Nicks, Chambon & Coombes, PRE 2018

- Key point – estimate the change in event time by linearizing around the boundary h

$$\delta T = -\frac{\nabla_{\mathbf{z}} h(\mathbf{z}(T^-)) \cdot \delta \mathbf{z}(T^-)}{\nabla_{\mathbf{z}} h(\mathbf{z}(T^-)) \cdot \dot{\mathbf{z}}(T^-)}$$

- Estimate the velocity using the velocity of the synchronous manifold

$$\dot{\tilde{\mathbf{z}}}(T^-) \simeq \dot{\mathbf{z}}(T^-)$$

Extensions of the MSF approach - discontinuities

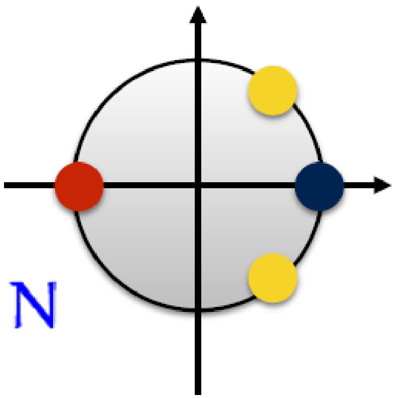
What does it all look like? Planar example:

$$\beta_l = \sigma \lambda_l \in \mathbb{C}$$

$$\dot{\xi} = [\mathbf{DF}(\mathbf{s}) - \beta_l \mathbf{DH}(\mathbf{s})]\, \xi, \qquad \xi \in \mathbb{R}^2$$

eigenvalues of $\mathcal{G}$

… modified Floquet problem.



Non smooth result: $\xi(\Delta) = \Gamma(l)\xi(0)$

$$\Gamma(l) = K_L G_L(l) K_R G_R(l) \in \mathbb{R}^{2\times 2}, \qquad l = 1, \ldots, N$$

$$G_\mu(l) = G(A_\mu - \beta_l J; \Delta_\mu), \qquad K_\mu = K(T_\mu), \qquad \mu \in \{L, R\}$$

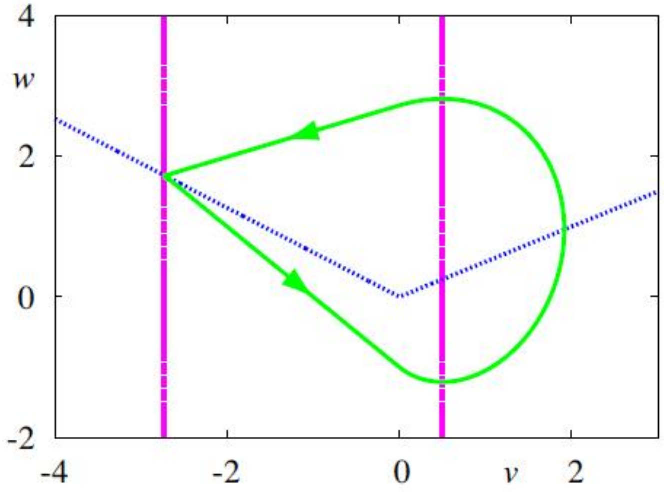
$$K(t) = \begin{bmatrix} \dot{v}(t^+)/\dot{v}(t^-) & 0 \\ (\dot{w}(t^+) - \dot{w}(t^-))/\dot{v}(t^-) & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Saltation matrix

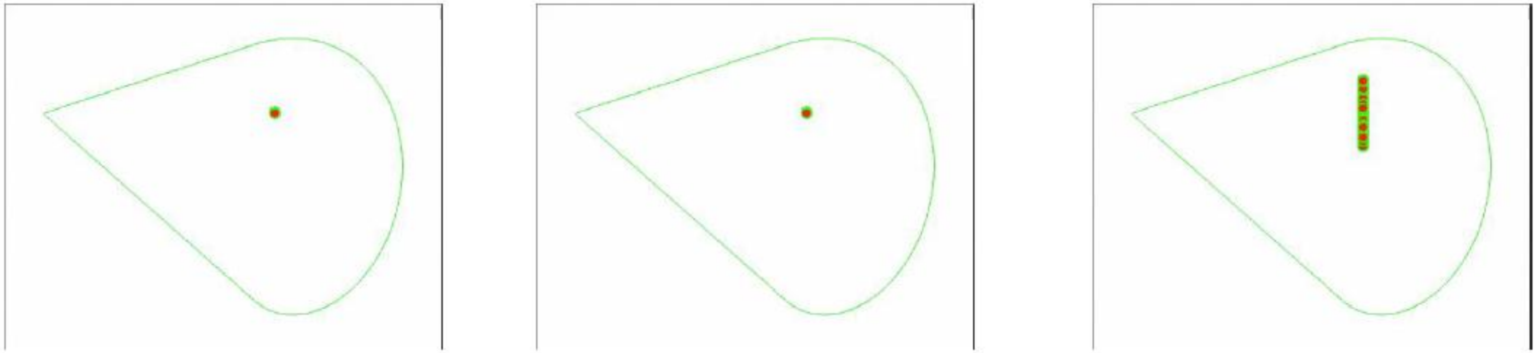
[coupling on the voltage variable]

Extensions of the MSF approach - discontinuities

Network of *homoclinic* oscillators



Global linear coupling on "v"

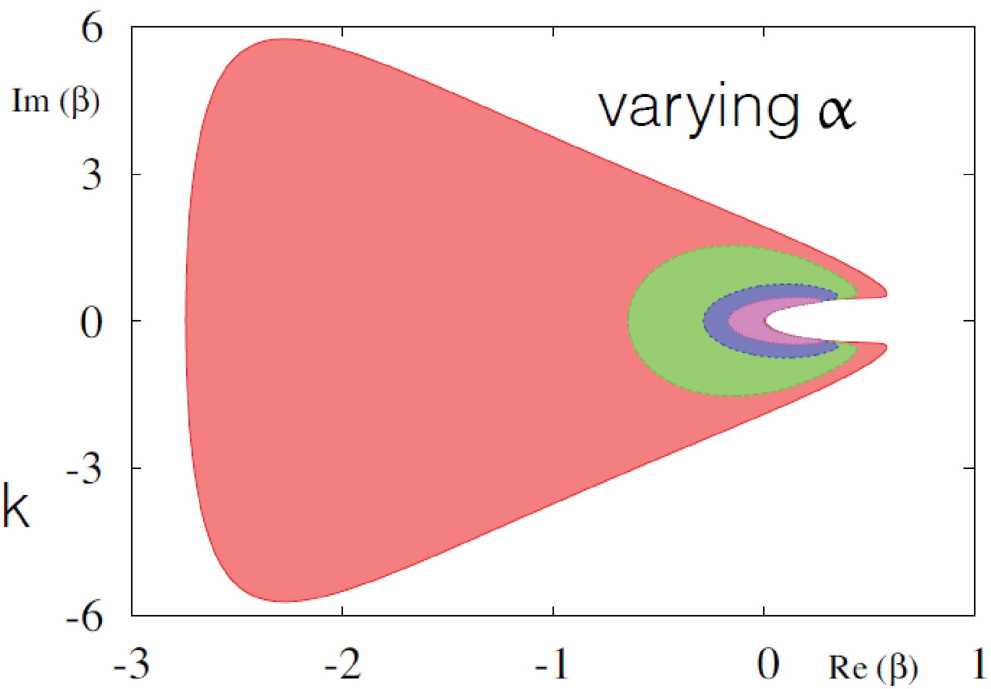


Synchrony unstable for weak coupling and restabilises via an inverse period doubling bifurcation at $\epsilon = \epsilon_{pd}$ in excellent agreement with simulations (independent of $\mathbf{N}$).

Coombes & Thul EJAM 2016

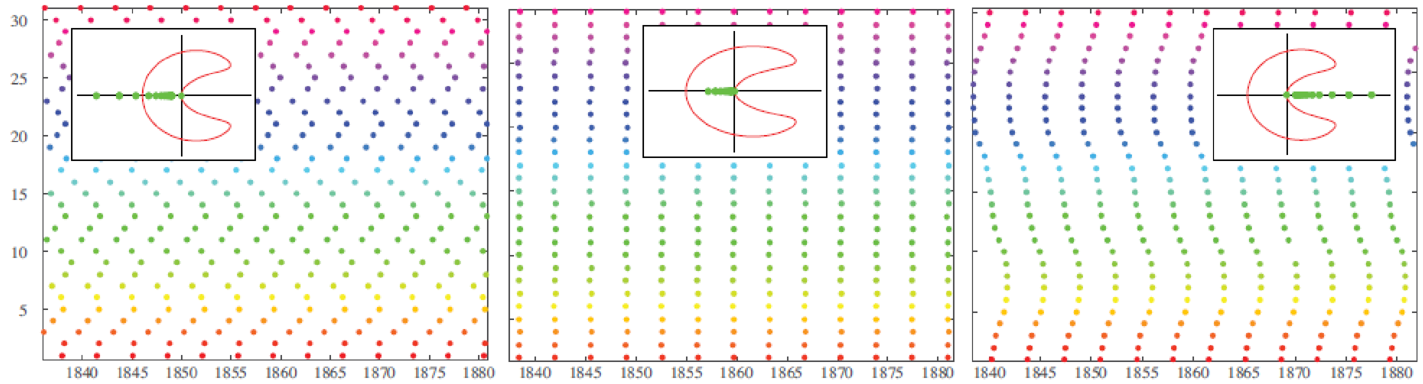
Extensions of the MSF approach - discontinuities

MSF: $\Gamma_l = K(\Delta)\exp\{(A_R + \sigma\lambda_l D\mathbf{H})\Delta\}$



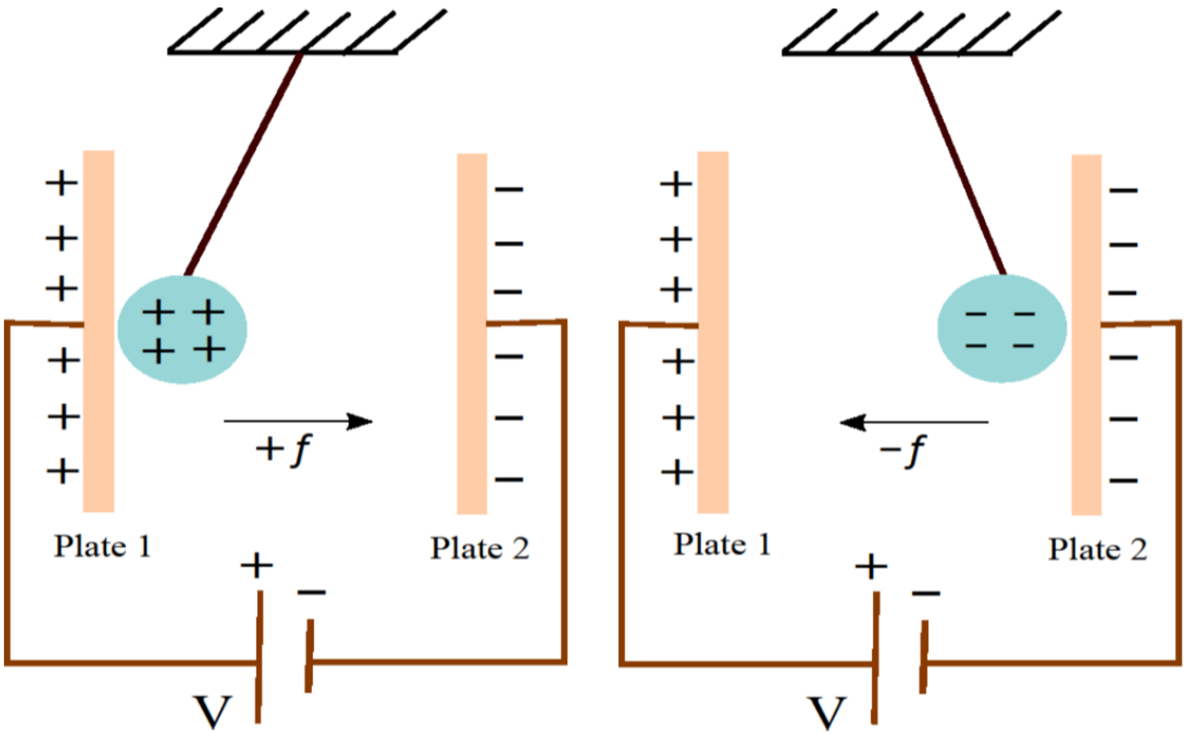
$$\beta = \sigma\lambda_l$$

Mex-hat
ring network
(N=31)
[circulant]



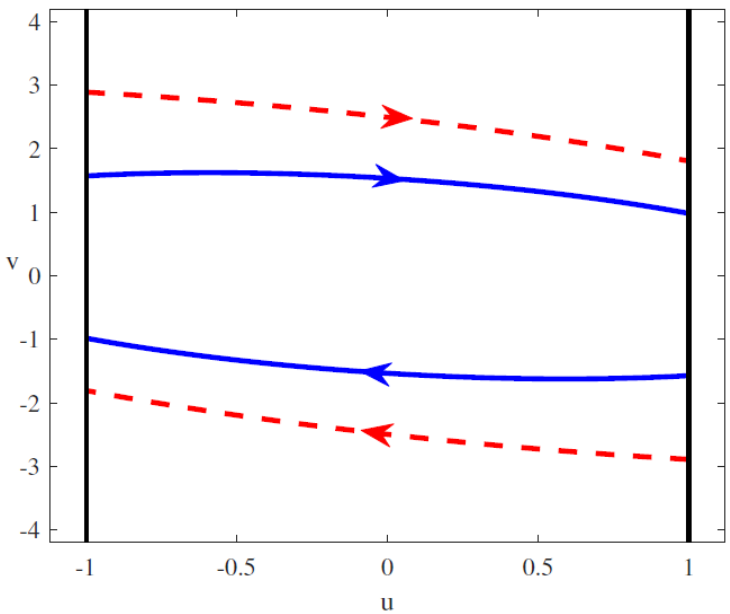
Coombes & Thul EJAM 2016

The Franklin Bell



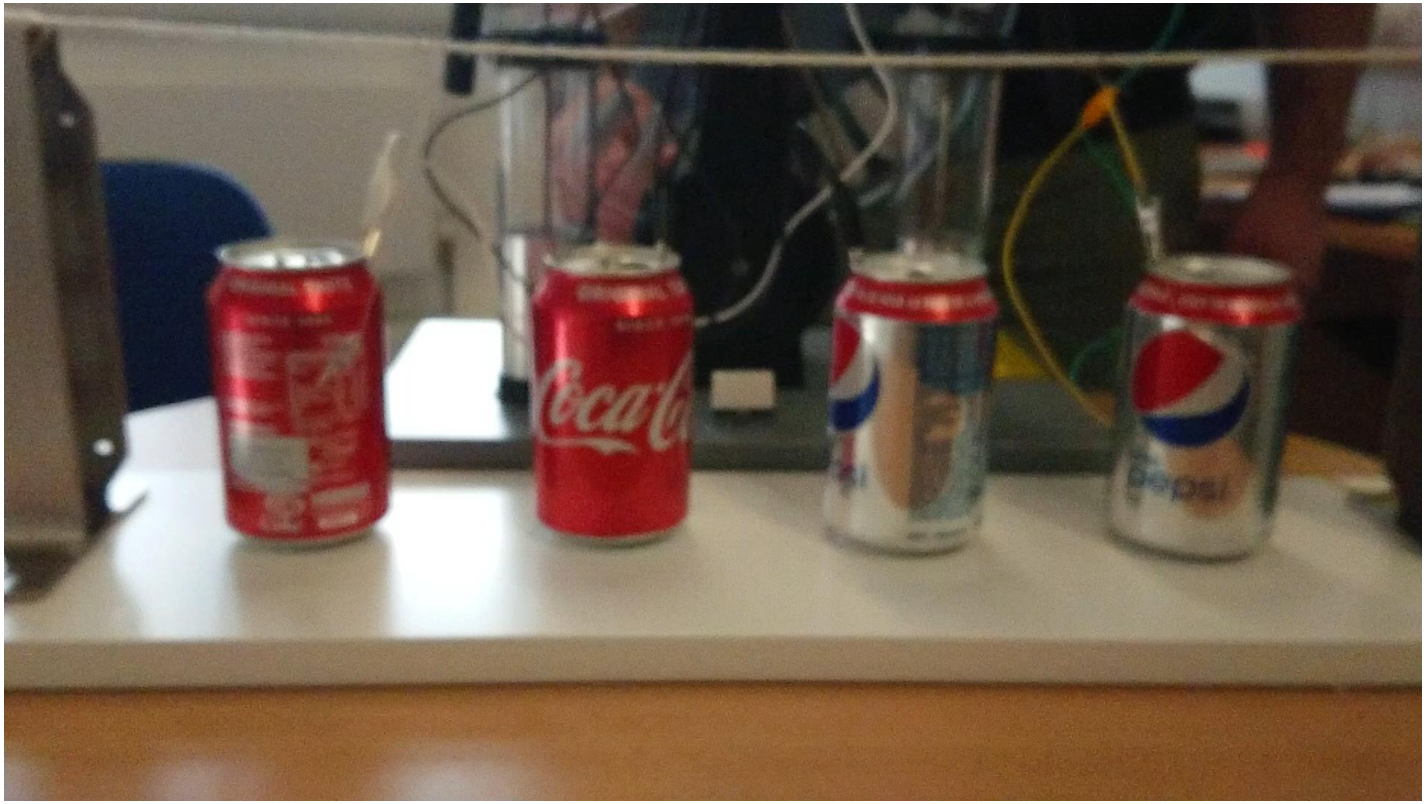
$$\ddot{u} + \gamma_1 \dot{u} + \gamma_2 u = \text{sgn}(\dot{u}) f, \qquad \text{if } t \neq t_i,$$
$$\dot{u}(t_i^+) = -k\dot{u}(t_i^-), \qquad \text{if } t = t_i.$$

Limit cycles

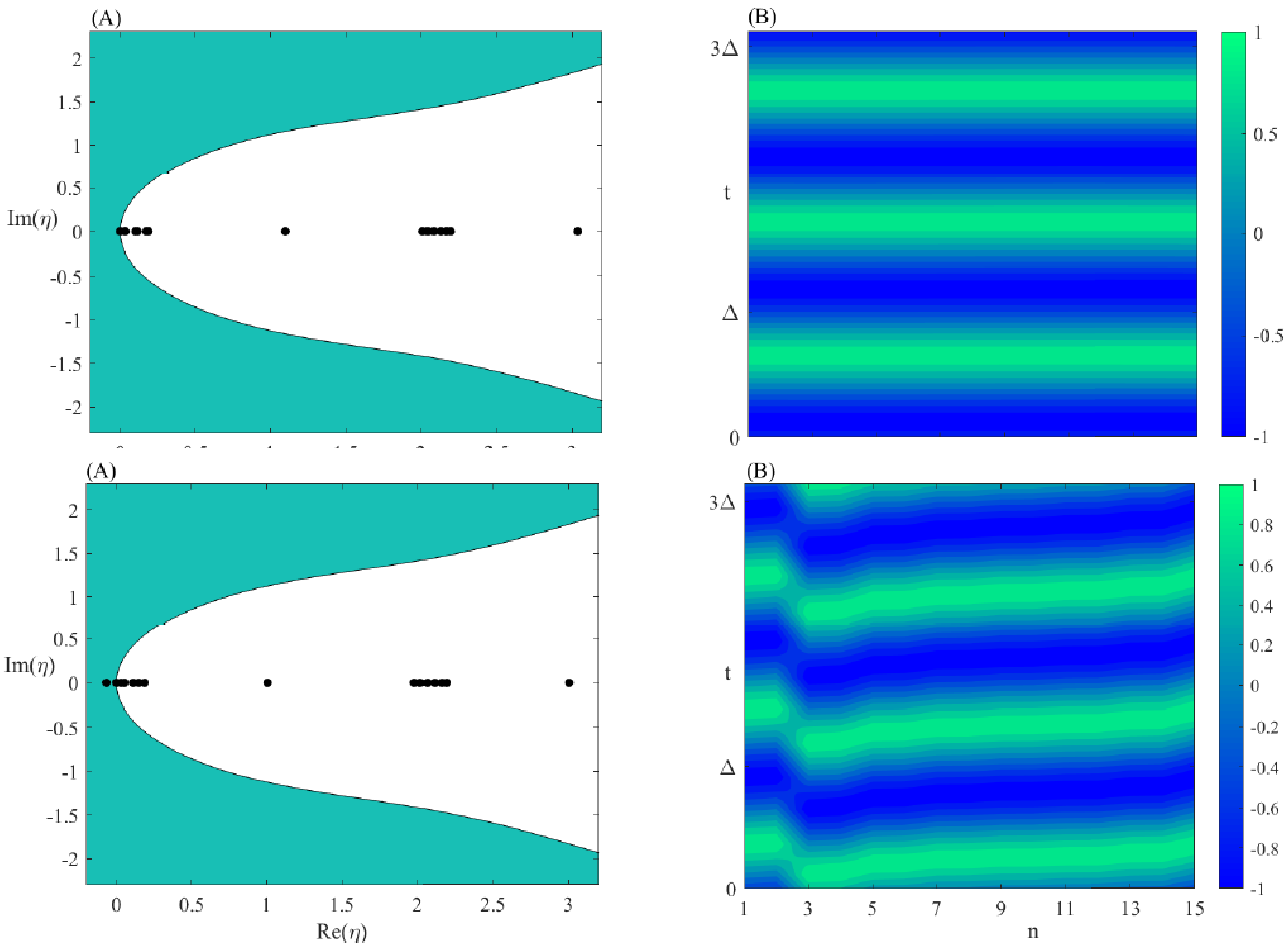


Şayli, YML, Thul & Coombes, under review

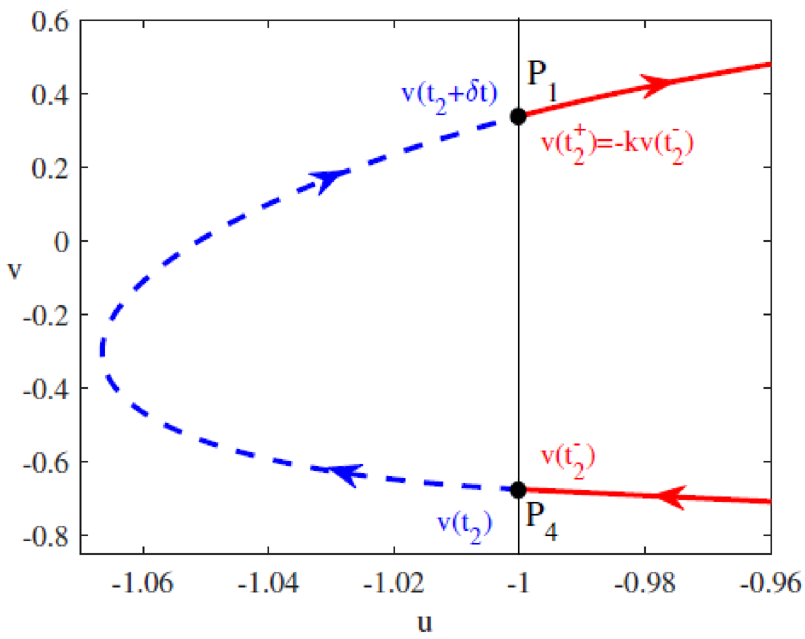
The Franklin Bell



The Franklin Bell – MSF analysis

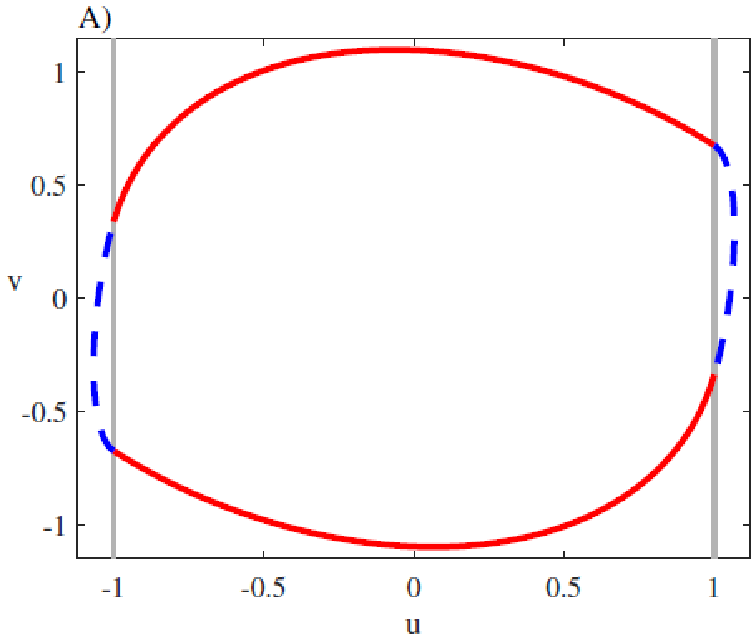


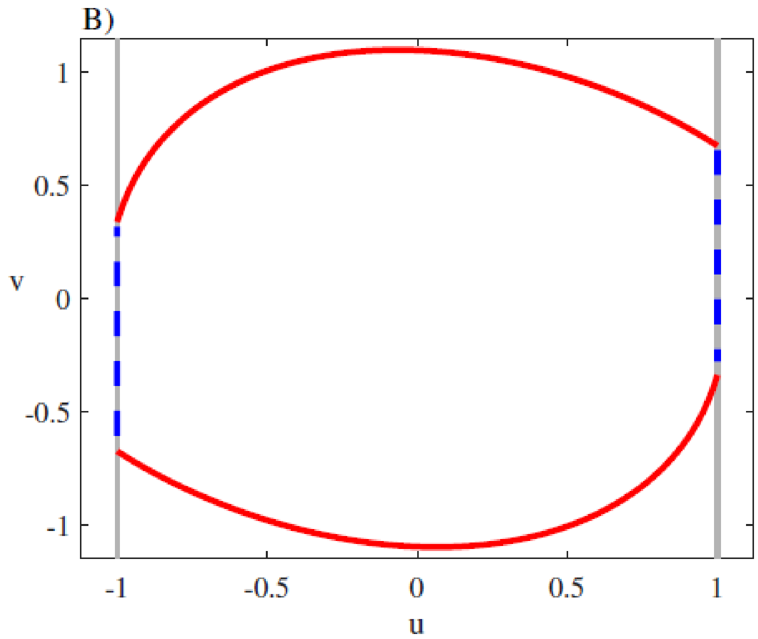
New method for smoothing



$$\frac{dx}{dt} = \begin{cases} Ax + f_e, & \text{if } |u| \leqslant a \\ A_R x + f_R, & \text{if } u > a \\ A_L x + f_L, & \text{if } u < -a \end{cases},$$

6 equations for 6 unknowns, controlled by time-of-flight parameter



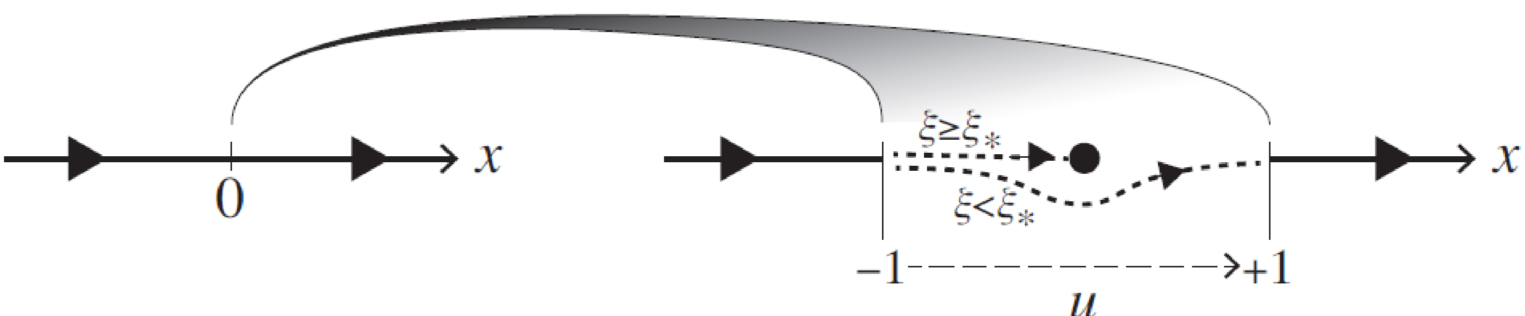


Nonsmooth systems: indeterminacy

Some nonsmooth systems can have indeterminacy at thresholds e.g. pausing

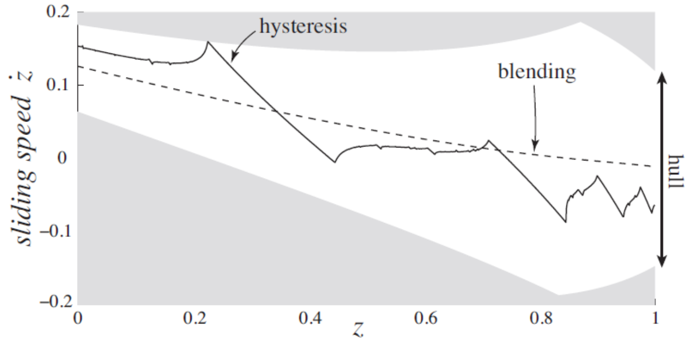
$$\dot{x} = \sqrt{|x|}\ , \qquad x(0) = 0$$

$$\phi_\tau(t) = \begin{cases} \phi_-(t) & \text{if } t < 0\ , \\ 0 & \text{if } 0 \leq t \leq \tau\ , \\ \phi_+(t-\tau) & \text{if } t > \tau\ , \end{cases}$$



Weber, Glendinning, Jeffrey,
Proc Roy Soc A 2019

Nonsmooth systems: indeterminacy

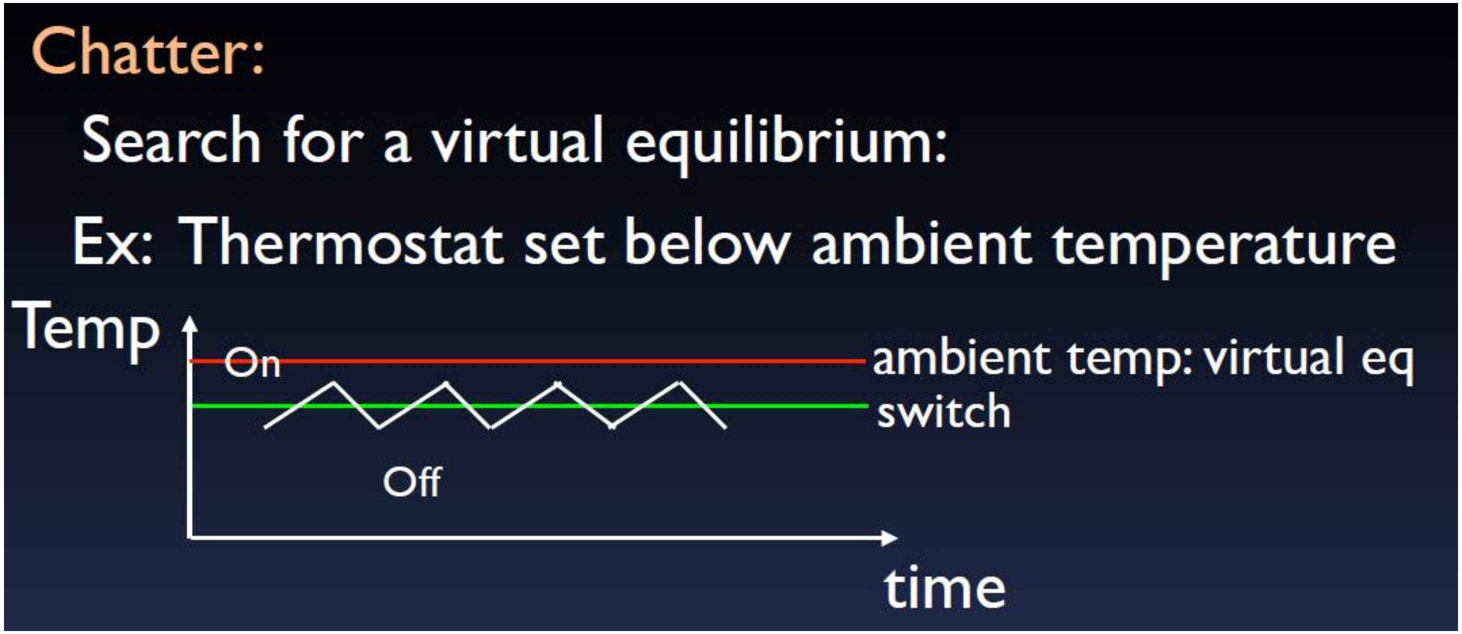


Some nonsmooth systems can have indeterminacy at thresholds e.g. pausing

Or other issues like jitter or chatter:

Jitter while sliding along a threshold

Jeffrey, Kafanas, Simpson, IJBC 2018



Nonsmooth systems: indeterminacy

Motivating example: optimization for large sparse problems e.g. seismic data

Non-smooth dynamics in optimization algorithms:

Example: look for solution of Ax=b,

Linearized Bregman (LB):

$$
\begin{aligned}
z_{k+1} &= z_k - t_k A^\top (A x_k - b) \\
x_{k+1} &= S_\lambda(z_{k+1}), \qquad S_\lambda(z_k) = \max(|z_k| - \lambda, 0)\,\mathrm{sign}(z_k)
\end{aligned}
$$

Yin, et al, 2008

S_λ is a shrinkage or thresholding operator - removes elements below threshold λ

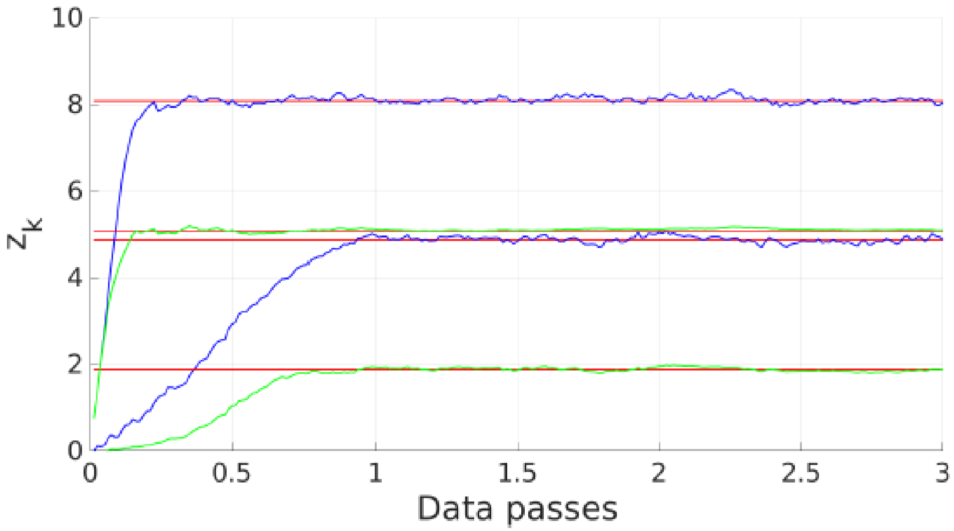
Kuske, presentation at SIAM DS19

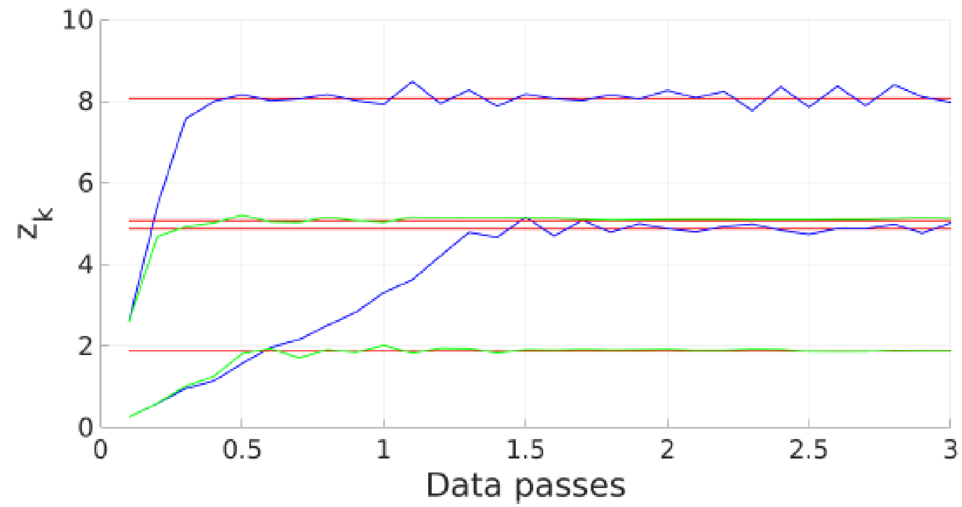
Nonsmooth systems: indeterminacy

One solution – time-step adjustment

$$
\begin{aligned}
z_{k+1} &= z_k - \tau_k \odot A_k^\top (A_k x_k - b_k) \\
x_{k+1} &= S_\lambda(z_{k+1}),
\end{aligned}
$$

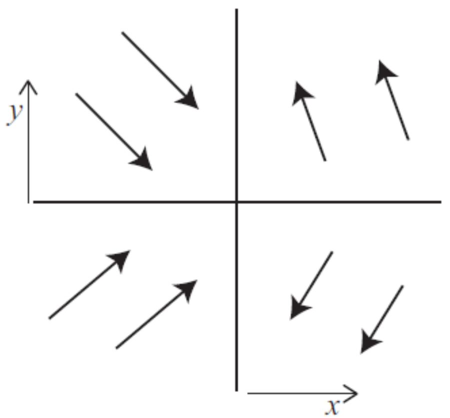
$$
\tau_k[i] = t_k \frac{\left| \sum_{j=1}^{k} \mathrm{sign}([A_j^\top (A_j x_j - b_j)]_i) \right|}{k}
$$





Kuske, presentation at SIAM DS19

Hidden dynamics



Another solution:

Add a “hidden layer” to resolve this

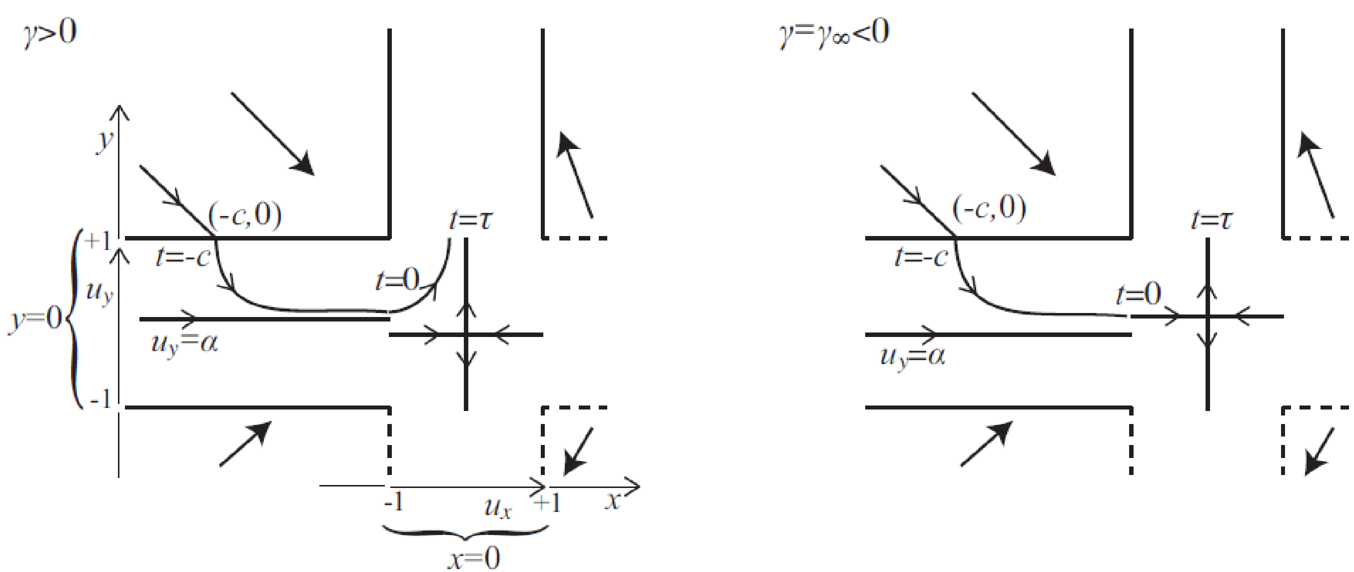


FIGURE 3. Geometry of the double switch, showing a solution arriving at $y = 0$, evolving through the interval $u_y \in (-1, +1)$, then through the double-interval $(u_x, u_y) \in (-1, +1) \times (-1, +1)$ in which there is a saddlepoint, and exiting into $y > 0$ at $t = \tau$.

Weber, Glendinning, Jeffrey, Proc Roy Soc A 2019

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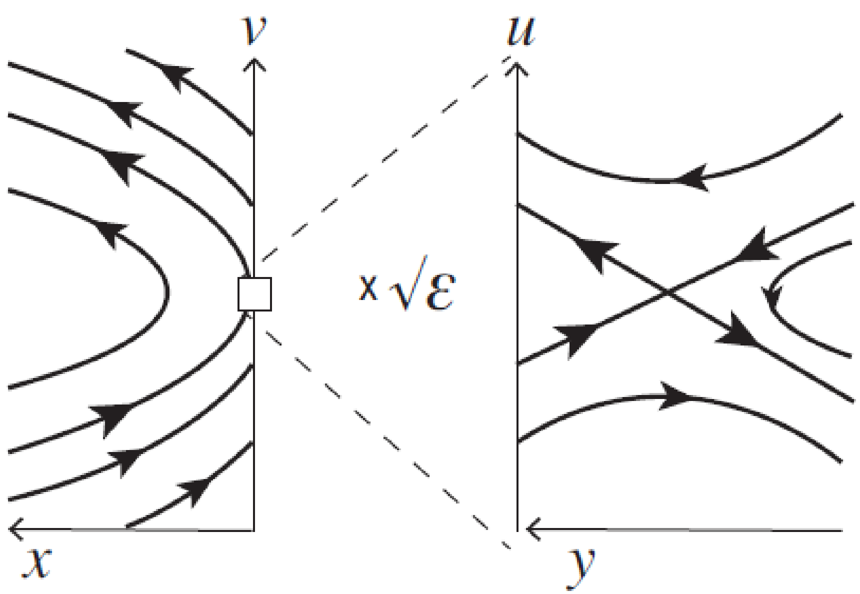
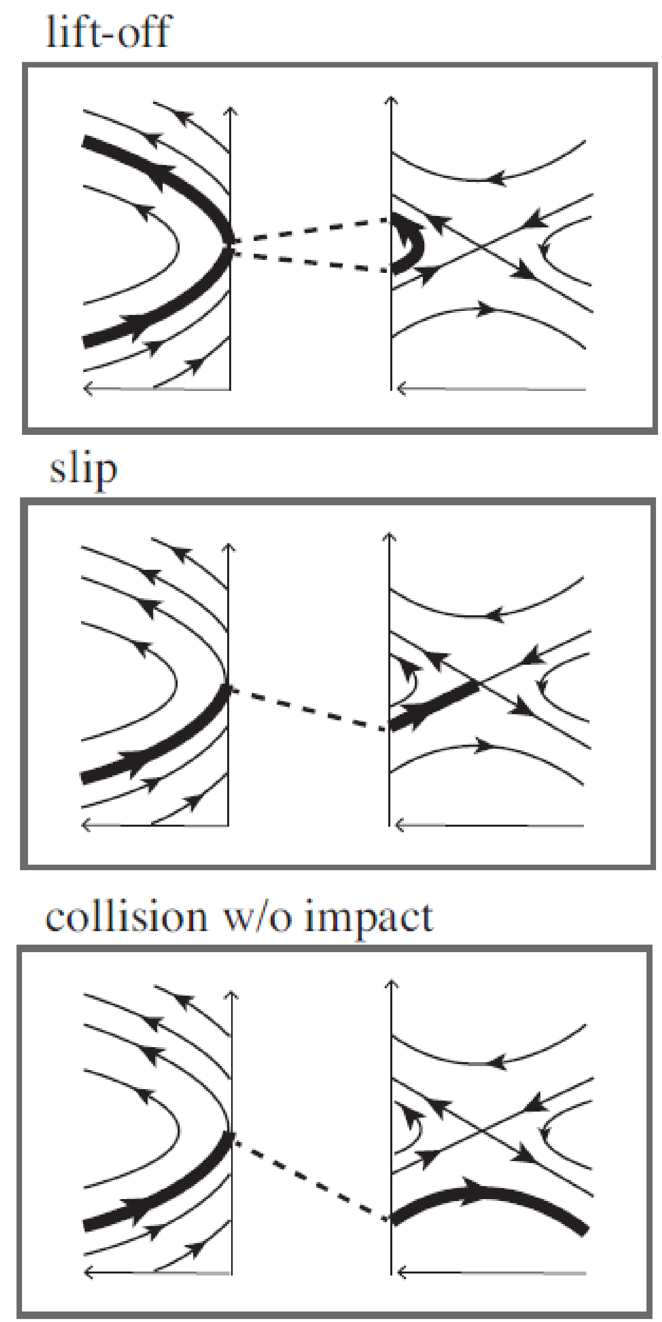
Hidden dynamics

Painleve’s paradox (indeterminacy in stick-slip systems)



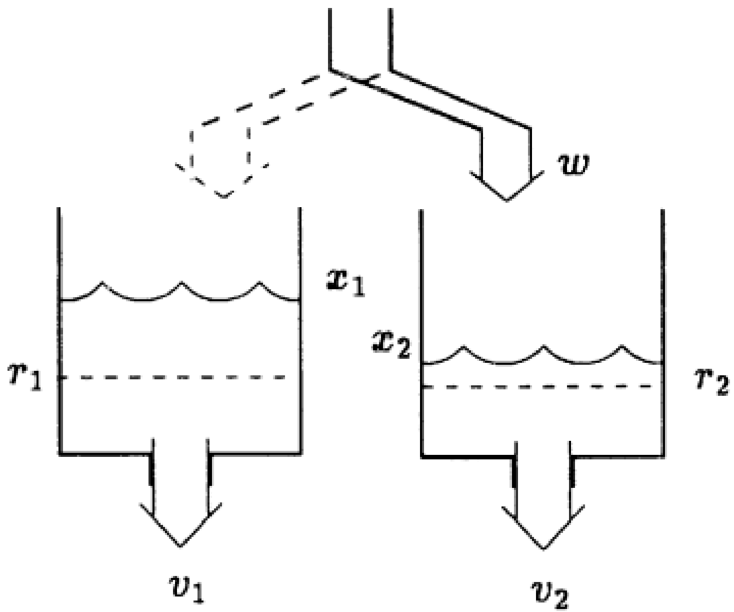
Weber, Glendinning, Jeffrey, Proc Roy Soc A 2019

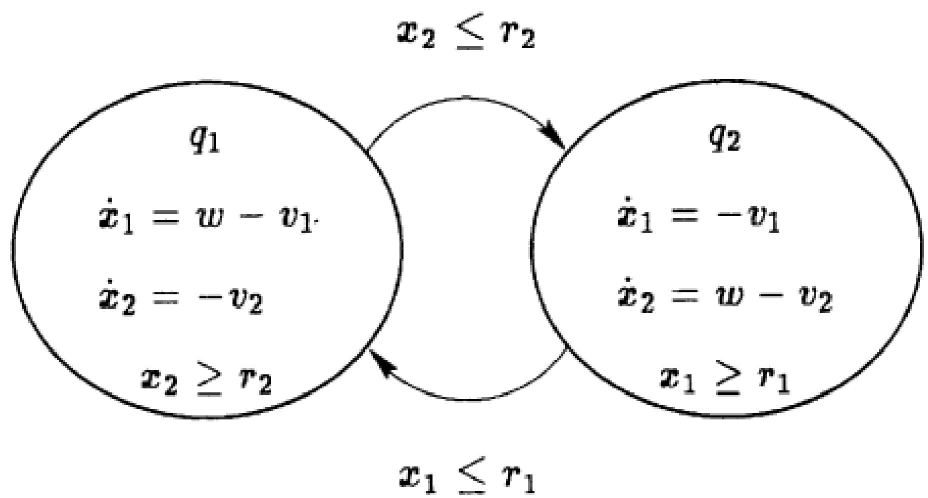


Zeno regularization

Another method: regularization via convolution with a probability distribution (Belgacem, Bensalah, Cherki, Edwards, SIAM DS 19 presentation)

What is a Zeno system?

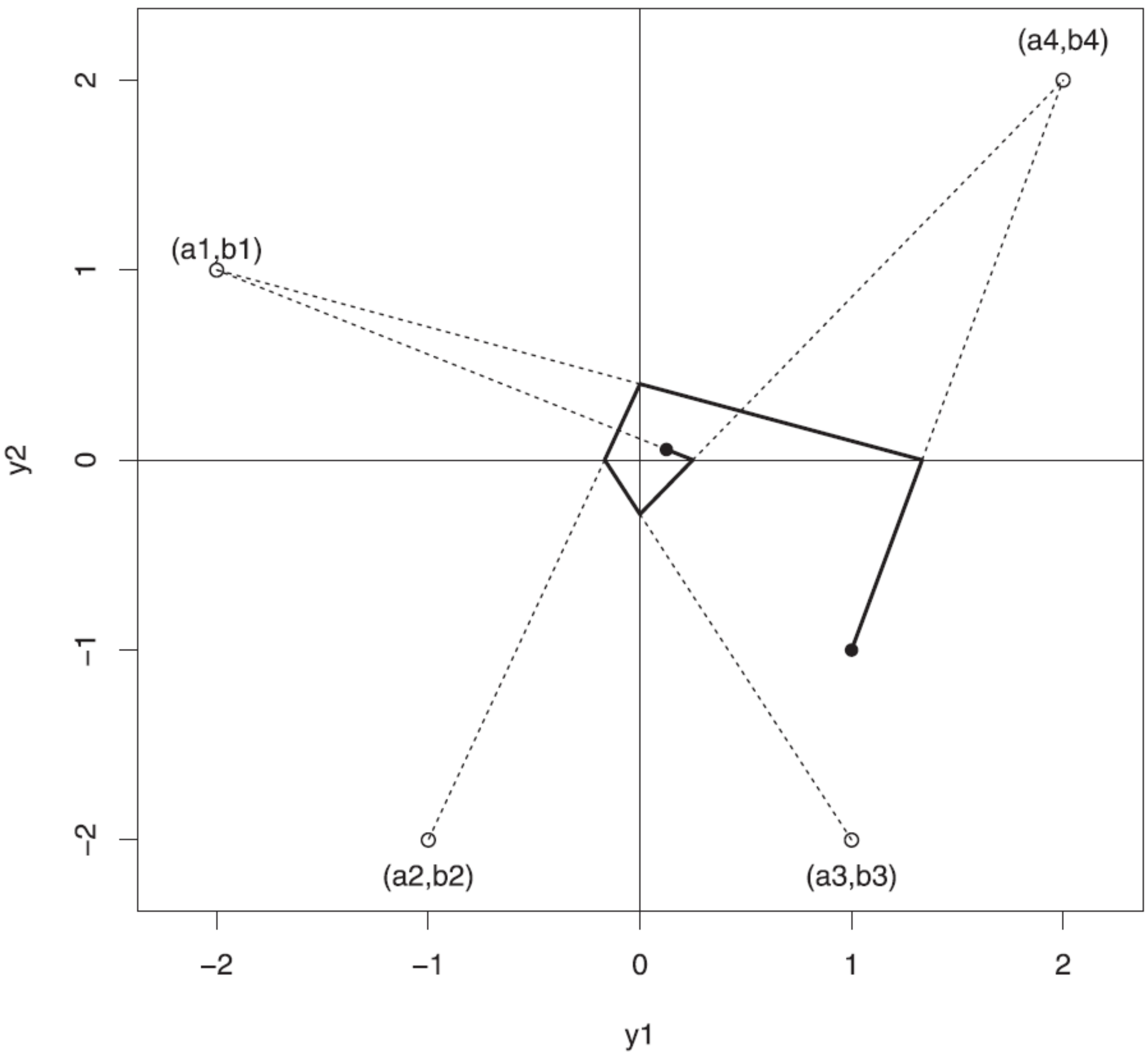




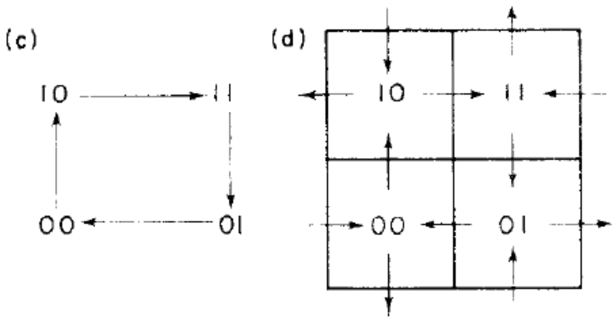
Switching water tank

Figure from:
Johansson, Egerstedt, Lygeros, Sastry, Syst Control Lett 1999

Zeno regularization



Damped oscillations in a two-gene network



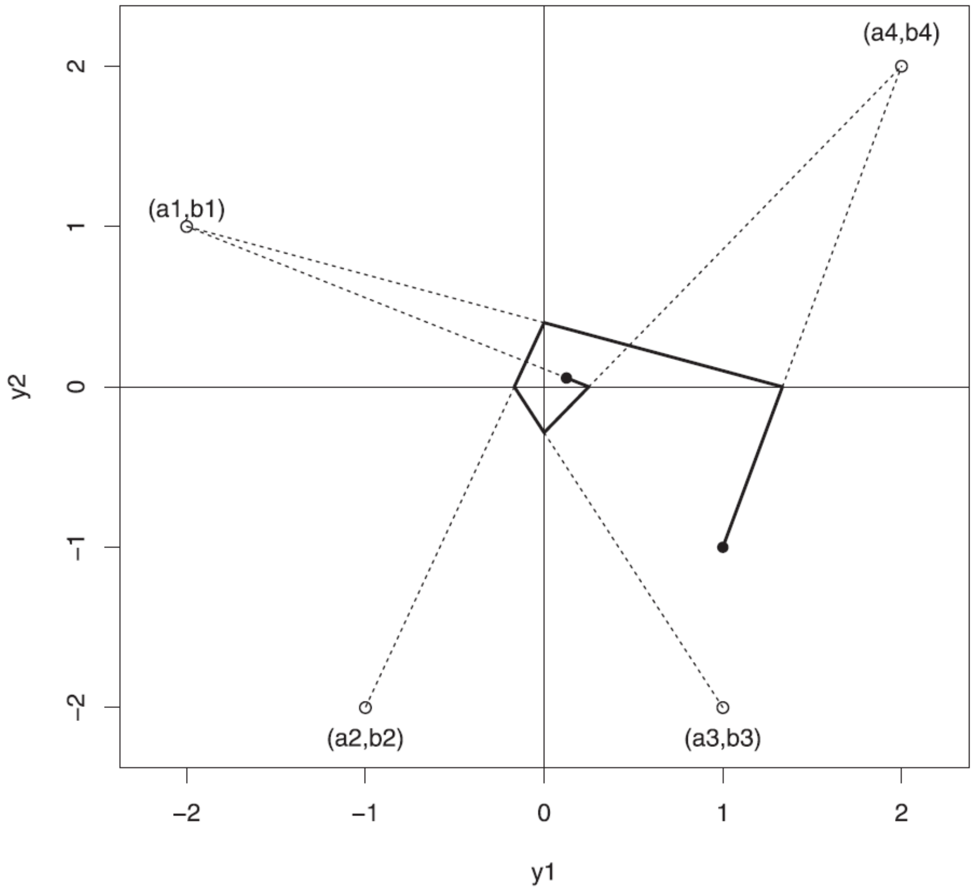
Edwards, EJAMS 2018

Summary – MSF for piecewise linear systems

	Continuous trajectories/ vector fields	Discontinuous trajectories / vector fields
Continuous interactions	A Matrix exponentials	B Matrix exponentials with saltation matrices
Discontinuous interactions	C Glass networks	D Ordering problem

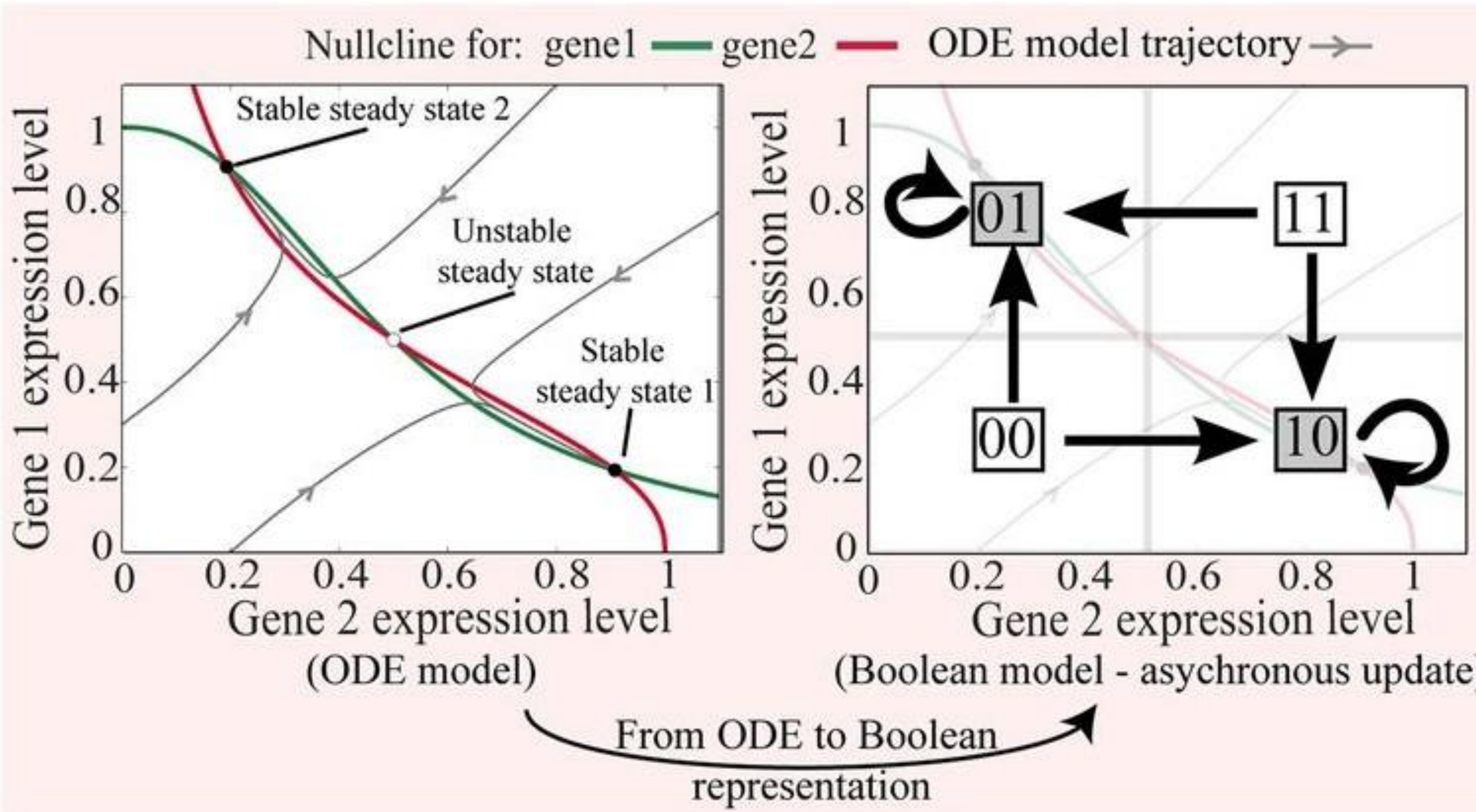
Glass networks

C



Glass networks

Middleton, Farcot, Owen, Ventoux, Plant Cell 2012

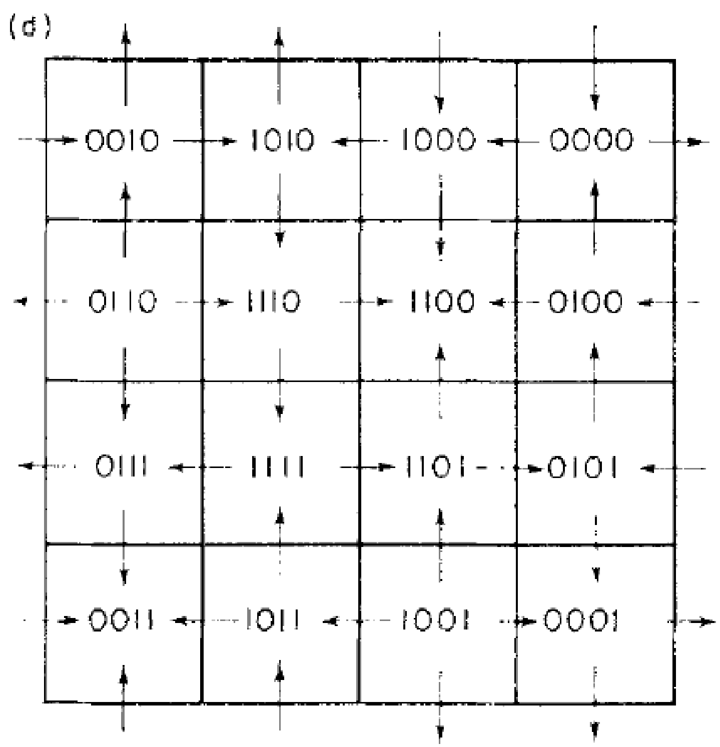


Glass networks

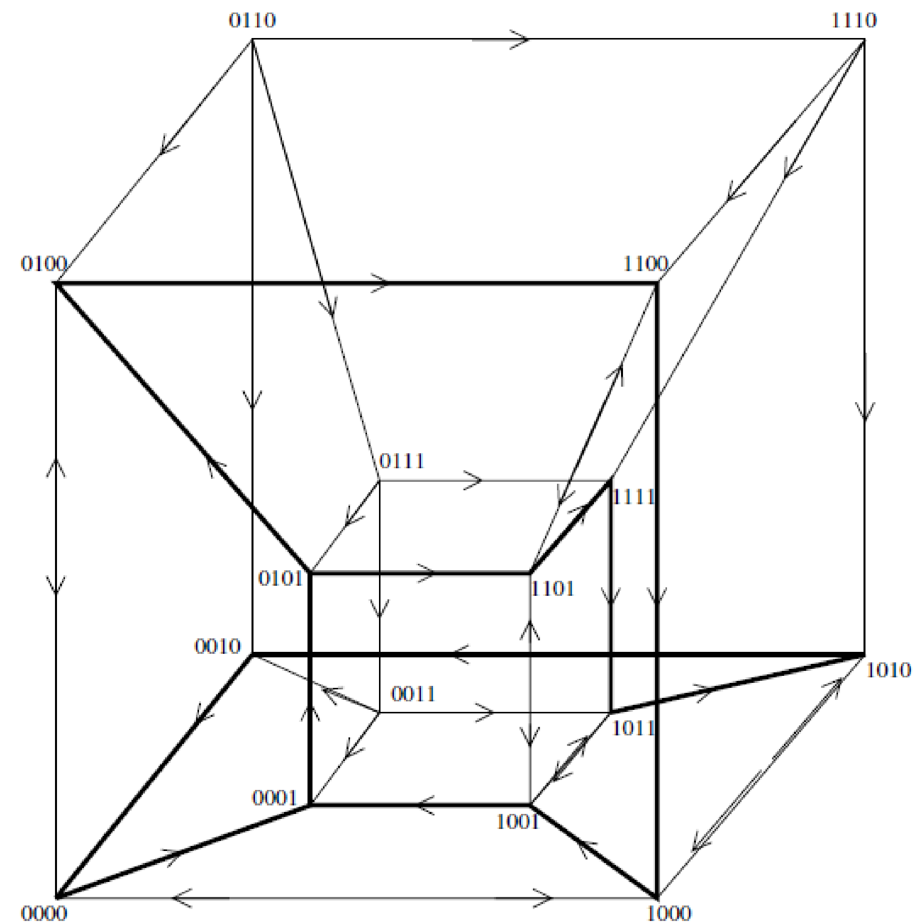
$$\dot{y}_i = -y_i + F_i(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n), \quad i = 1, \ldots, n,$$

where

$$\tilde{y}_i = \begin{cases} 0 & \text{if } y_i < 0, \\ 1 & \text{if } y_i > 0. \end{cases}$$



Glass & Kaufman, J Theor Biol 1973



Lu & Edwards, Int J Bif Chaos 2011

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Summary – MSF for piecewise linear systems

	Continuous trajectories/ vector fields	Discontinuous trajectories / vector fields
Continuous interactions	A Matrix exponentials	B Matrix exponentials with saltation matrices
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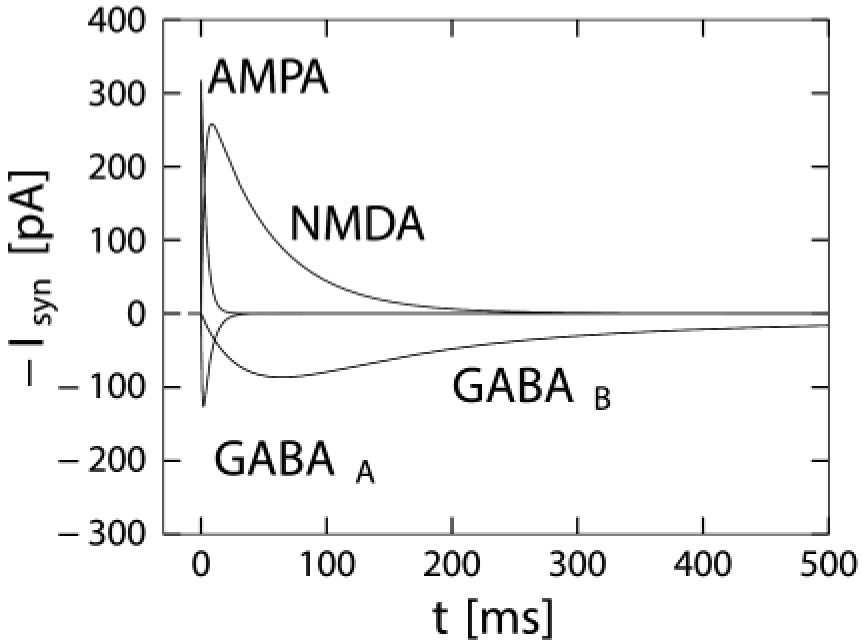
Adding discontinuous interactions

D

- Motivating example: a linear IF neuron

$$\frac{\mathrm{d}}{\mathrm{d}t} v_i = -\frac{v_i}{\tau_m} + I + I_i(t)$$

- External input from the network is modulated by synapses
- Consider exponential synapses
- The synaptic variable decays exponentially
- When a spike occurs it is incremented *instantaneously*
- dv/dt can change by large amounts during a set of crossings
- *Ordering* problem



Neuronal Dynamics, Gerstner et al.

Toy example: Two neurons

Consider a pair of IF neurons with exponential synapses

$$\frac{dv_1}{dt} = \mathrm{I} - \frac{v_1}{\tau} + \epsilon(s_2 - s_1)$$

$$\frac{dv_2}{dt} = \mathrm{I} - \frac{v_2}{\tau} + \epsilon(s_1 - s_2)$$

$$\frac{ds_1}{dt} = -\alpha s_1$$

$$\frac{ds_2}{dt} = -\alpha s_2$$

Spiking rule: when $v = V_{th}$

then $v \to V_r, s \to s + \alpha$

Differences:

$$\frac{d}{dt}\Delta v = -\frac{\Delta v}{\tau} - 2\epsilon\Delta s$$

$$\frac{d}{dt}\Delta s = -\alpha\Delta s$$

Contracting!

The catch – when the synchronous state is perturbed slightly, spikes happen at some small δT apart, during which

$$v_1 \approx V_r, \qquad v_2 \approx V_{th},$$
$$s_2 - s_1 \approx -\alpha$$

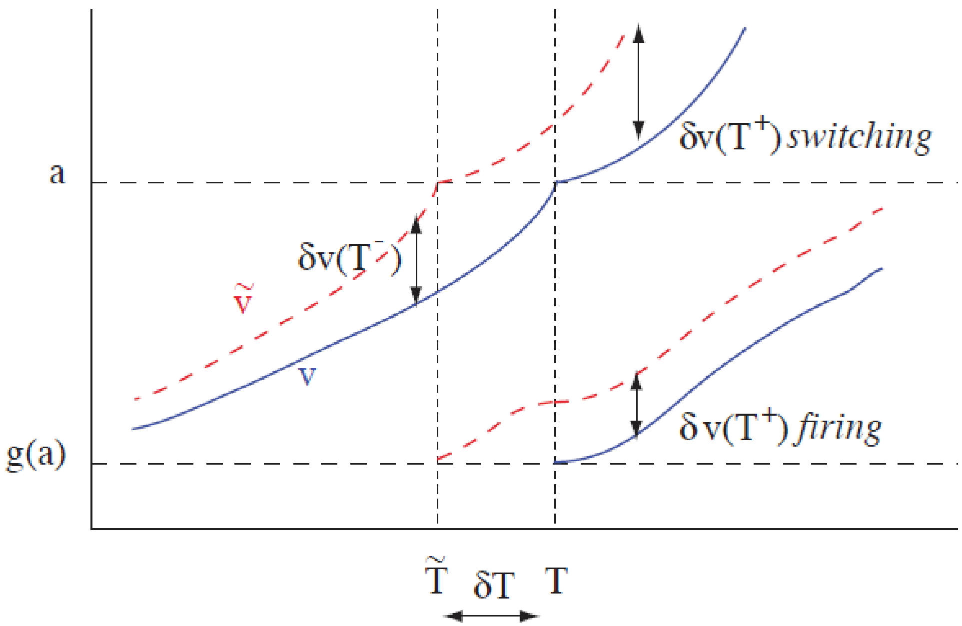
We found the synchronous state to be *unstable* for $\epsilon > 0$

Recap

- For discontinuous trajectories, this approach has to be complemented by saltation matrices
- Evolution of perturbations through a discontinuity
  - Key point – estimate the change in event time by linearizing around the boundary h

$$\delta T = -\frac{\nabla_{\mathbf{z}} h(\mathbf{z}(T^-)) \cdot \delta \mathbf{z}(T^-)}{\nabla_{\mathbf{z}} h(\mathbf{z}(T^-)) \cdot \dot{\mathbf{z}}(T^-)}$$

  - Estimate the velocity using the velocity of the synchronous manifold



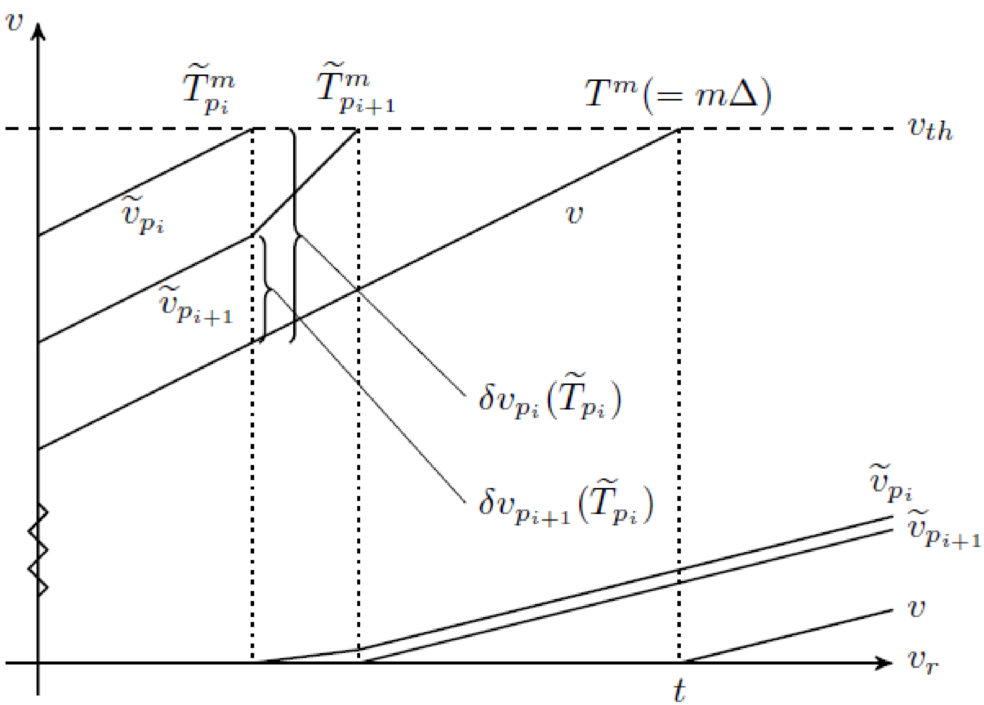
Nicks, Chambon & Coombes, PRE 2018

$$K(T) = Dg(\mathbf{z}(T^-)) + \frac{[\dot{\mathbf{z}}(T^+) - Dg(\mathbf{z}(T^-))\dot{\mathbf{z}}(T^-)]\,[\nabla_{\mathbf{z}} h(\mathbf{z}(T^-))]^T}{\nabla_{\mathbf{z}} h(\mathbf{z}(T^-)) \cdot \dot{\mathbf{z}}(T^-)}$$

$$= \begin{bmatrix} \dot{v}(T^+)/\dot{v}(T^-) & 0 & 0 & 0 \\ (\dot{w}(T^+) - \dot{w}(T^-))/\dot{v}(T^-) & 1 & 0 & 0 \\ (\dot{s}(T^+) - \dot{s}(T^-))/\dot{v}(T^-) & 0 & 1 & 0 \\ (\dot{u}(T^+) - \dot{u}(T^-))/\dot{v}(T^-) & 0 & 0 & 1 \end{bmatrix}$$

Network Saltation

This time, saltation needs to be handled for each crossing



- Crossing times need to be calculated iteratively

$$\delta T^m_{p_i} - \delta T^m_{p_{i+1}} = \frac{\delta v_{p_{i+1}}(\widetilde{T}^m_{p_i}) - \delta v_{p_i}(\widetilde{T}^m_{p_i})}{\widetilde{v}'_{p_{i+1}}(\widetilde{T}_{p_{i+1}})}$$

- Velocity terms have to be updated according to the order of crossings

$$\widetilde{v}'_{p_j}(\widetilde{T}^m_{p_{i+1}}) = \begin{cases} \widetilde{v}'_{p_j}(\widetilde{T}^m_{p_i}) + \alpha\sigma W_{p_j,p_i} & p_j \geq p_{i+1} \\ -v_r/\tau_m + I + \alpha\sigma[W_{p_j,p_j} + W_{p_j,p_i}] & p_j < p_{i+1} \end{cases}$$

~~$\dot{\mathbf{z}}(T^-) = \mathbf{z}(T^-)$~~

$$K_m(p_i) = \begin{cases} I_{2N} - \dfrac{\left(\widetilde{z}'(\widetilde{T}^m_{p_i}) - \overline{z}'(\widetilde{T}^m_{p_i})\right)e^{\mathsf{T}}_{p_i}}{\overline{v}'(\Delta^-)} & i = N \\ I_{2N} + \dfrac{(\widetilde{z}'(\widetilde{T}^m_{p_i}) - \overline{z}'(\widetilde{T}^m_{p_i})(e^{\mathsf{T}}_{p_i} - e^{\mathsf{T}}_{p_{i+1}})}{\overline{v}'(\Delta^-) + \alpha\sigma\sum_{j=1}^{i} W_{p_{j+1},p_j}} & i \neq N \end{cases}$$

Network Saltation

- The variational equations can no longer be diagonalized
- Propagation between crossings is still handled by a single (large – not block diagonalized) matrix exponential

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta z = J\delta z \qquad J = \begin{bmatrix} -I_N/\tau & \sigma W \\ 0_N & -\alpha I_N \end{bmatrix}$$

- Stability (over m periods):

$$\begin{aligned}\Gamma(m) = {} & K_m(p_N)\ldots K_m(p_1)G(\Delta)K_{m-1}(p_N)\ldots K_{m-1}(p_1) \\ & \times G(\Delta)\ldots K_2(p_N)\ldots K_2(p_1)G(\Delta)K_1(p_N)\ldots K_1(p_1)G(\Delta)\end{aligned}$$

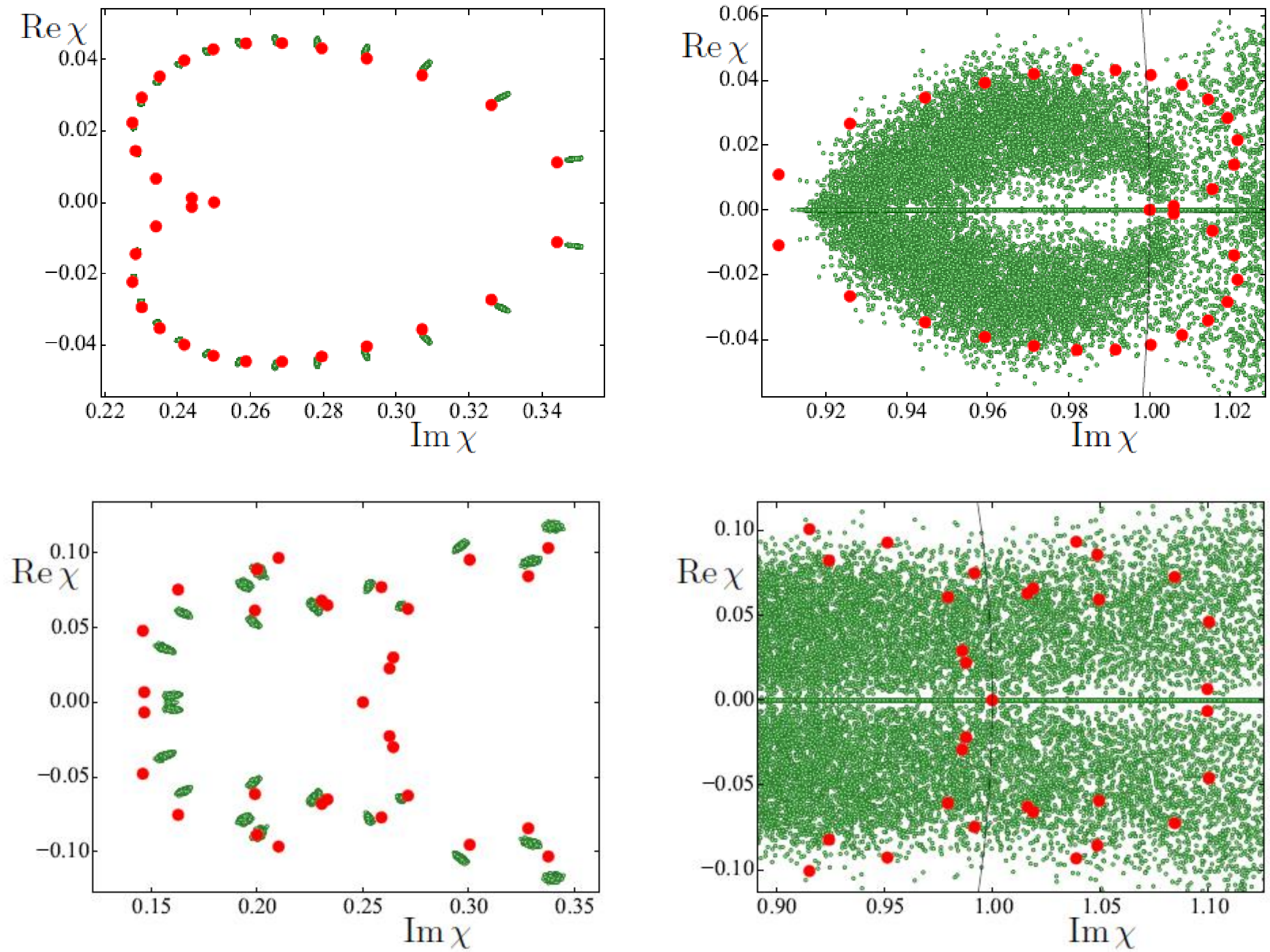
- In practice we consider some large number of random orderings or sets of initial conditions
- We reobtain the MSF if we use this approach for smooth synapses

Comparison to continuous synapse

- We compare against stability analysis for a continuous (but very fast) synapse
- Some eigenvalues are organized around those from the smooth system

Structured ring

Ring with random weights

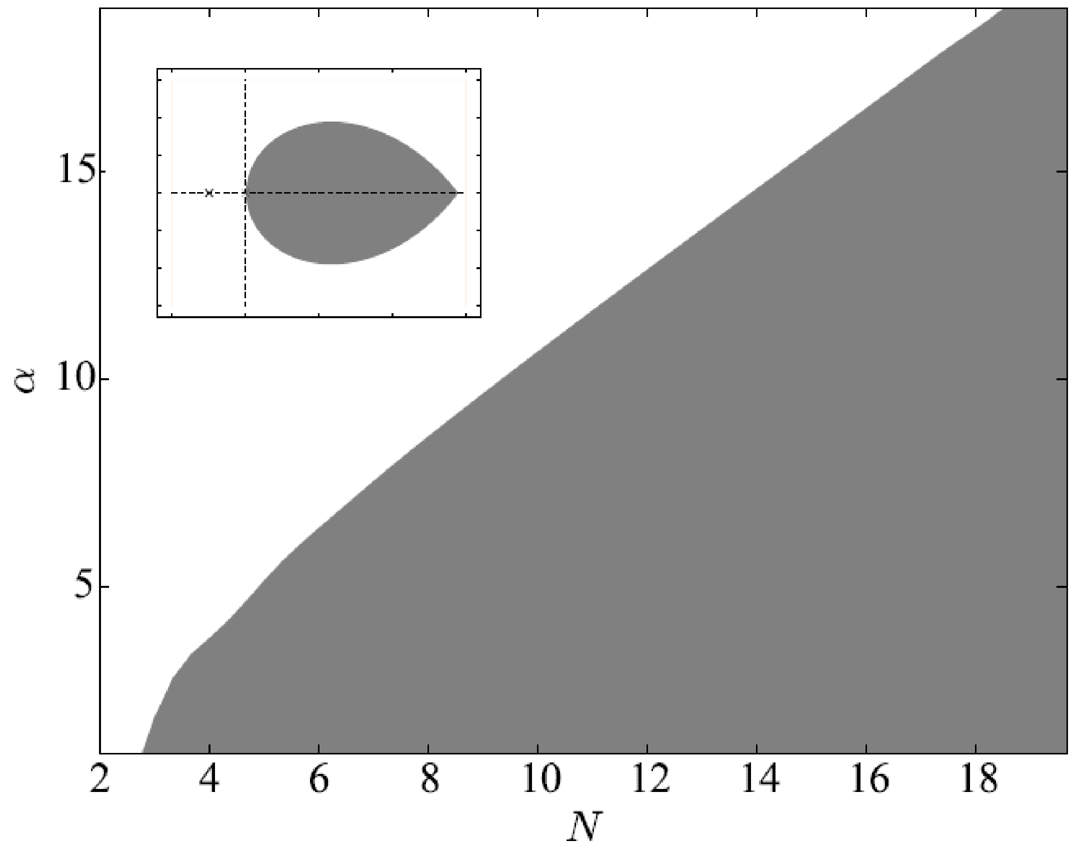


Lai, Thul, & Coombes, EPJ Special Topics 2018

Comparison to continuous synapse

- For global (all-to-all and identically weighted) networks, the degeneracy removes the combinatorial explosion
- We were able to sweep the parameter space (alpha = synapse speed, N = number of neurons)

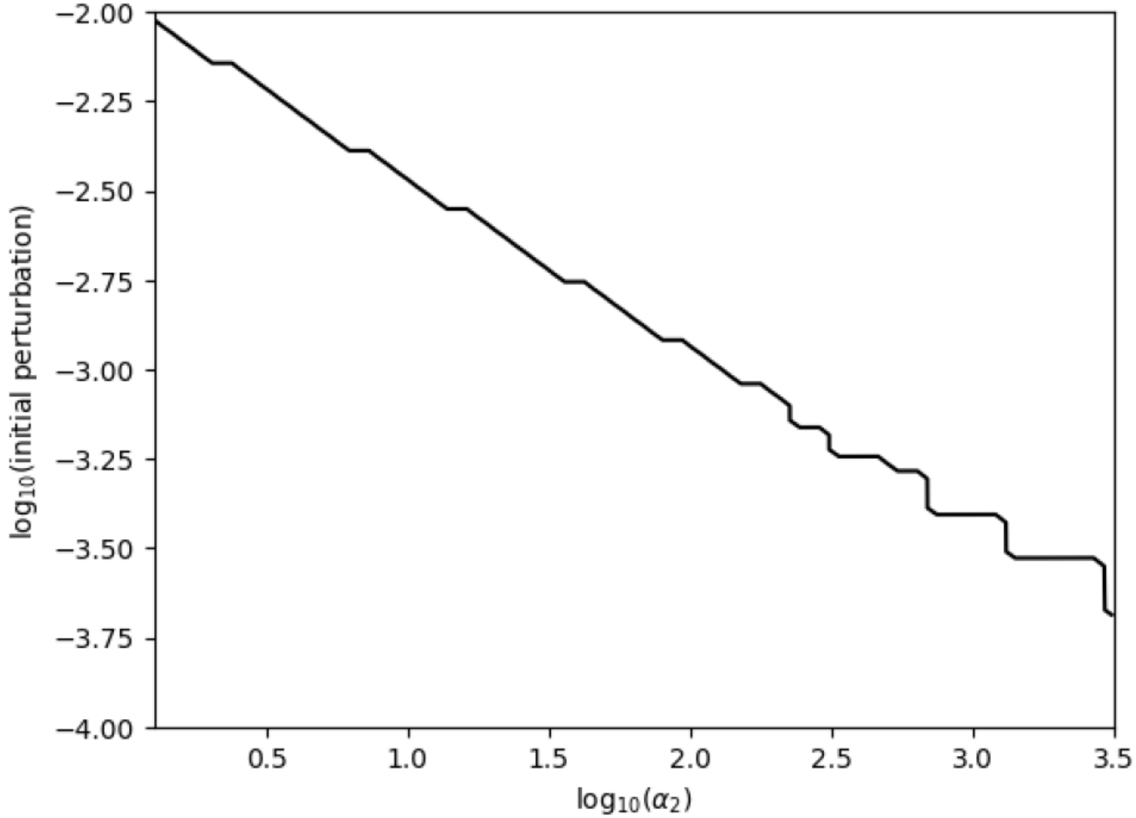
  - Synchrony can vary as a function of N even if the eigenvalues of the network stay the same
  - Goel & Ermentrout, Physica D (2002) find a similar phenomena in pulse-coupled phase oscillators
  - (Switching the sign of coupling gives a regime which is unstable for discontinuous synapses and stable for continuous synapses)



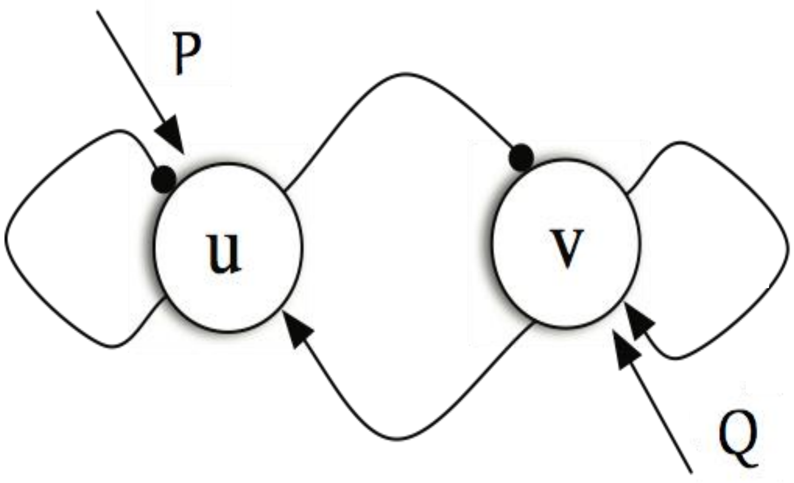
Lai, Thul, & Coombes, EPJ Special Topics 2018

Practical implications

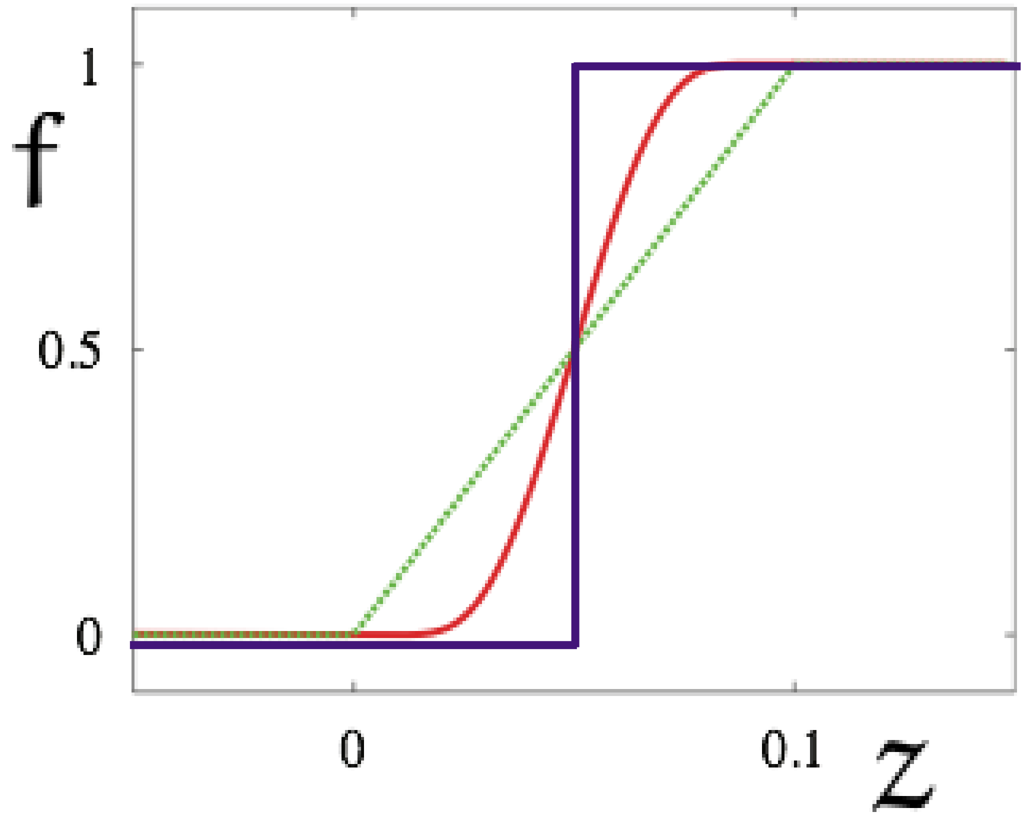
- We tried to explore this discrepancy with numerical simulations
- How would numerical simulations of a system “know” if it was genuinely nonsmooth or merely a very steep smooth synapse?
- We looked at a pair of coupled neurons and performed detailed numerical simulations for varying steepness and initial perturbation sizes
- In the regime where the nonsmooth synapse is unstable but the smooth approach predicts stability, we find a vanishingly small basin of attraction for synchrony
- Important to know relevant modelling scales!



Another example – Wilson-Cowan network

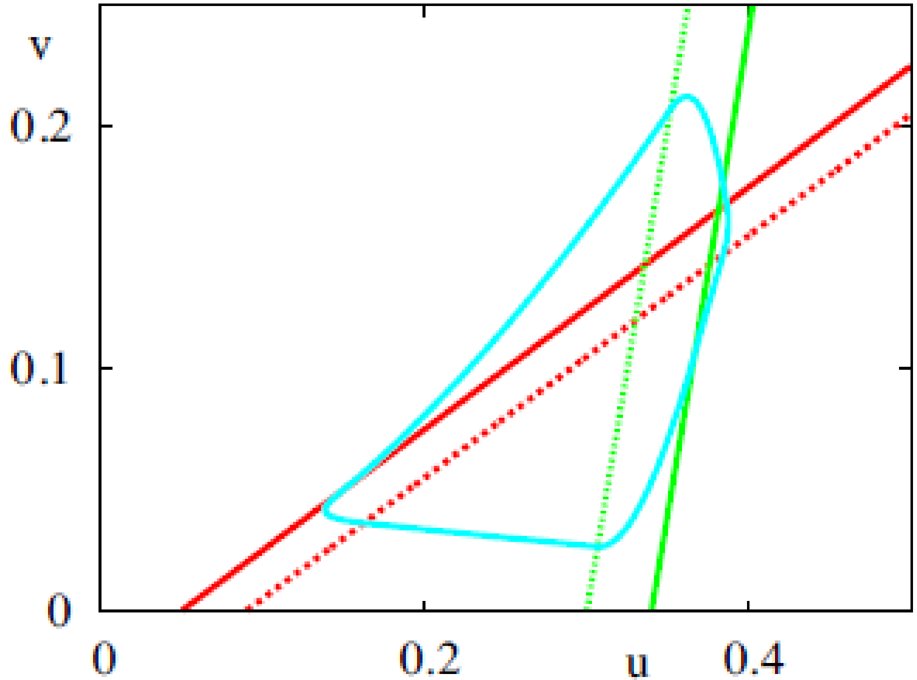


$$\dot{u} = -u + f(I_u + w^{uu}u - w^{vu}v),$$
$$\tau\dot{v} = -v + f(I_v + w^{uv}u - w^{vv}v)$$



Coombes, Lai, Şayli & Thul, EJAM 2018.

Piecewise linear Wilson-Cowan network

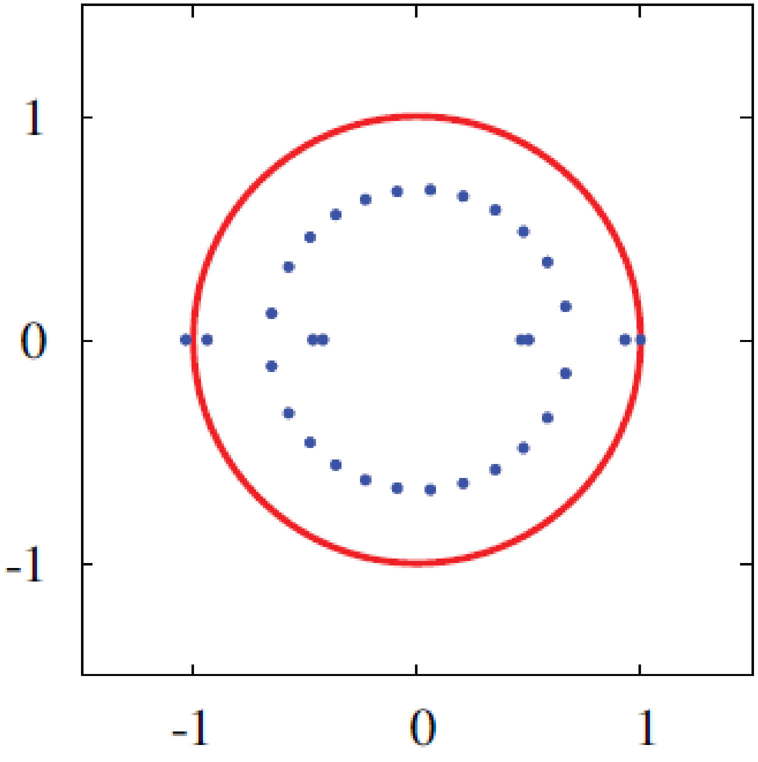


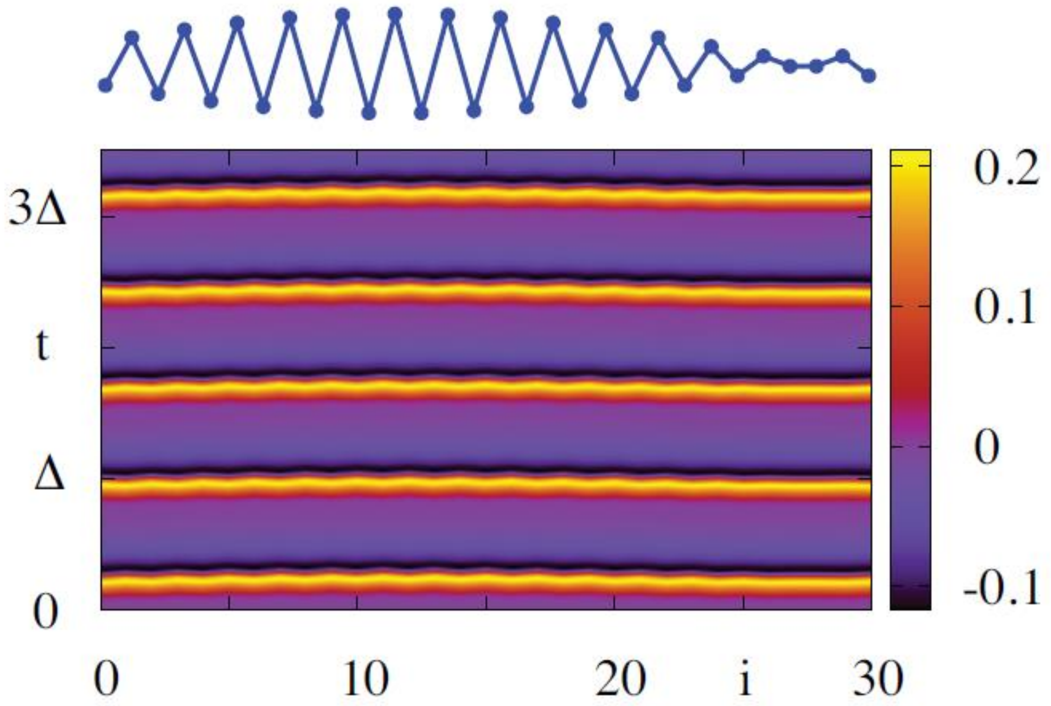
$$\Gamma(p) = \mathrm{e}^{A(p)\Delta_8}\mathrm{e}^{A_+(p;\epsilon)\Delta_7}\mathrm{e}^{A(p)\Delta_6}\mathrm{e}^{A_-(p;\epsilon)\Delta_5}\mathrm{e}^{A(p)\Delta_4}\mathrm{e}^{A_+(p;\epsilon)\Delta_3}\mathrm{e}^{A(p)\Delta_2}\mathrm{e}^{A_-(p;\epsilon)\Delta_1}$$

- Same ideas as before, except the propagation and saltation has to be performed at multiple thresholds
- As the vector field is continuous at thresholds, saltation is the identity matrix (like in the cardiac model)

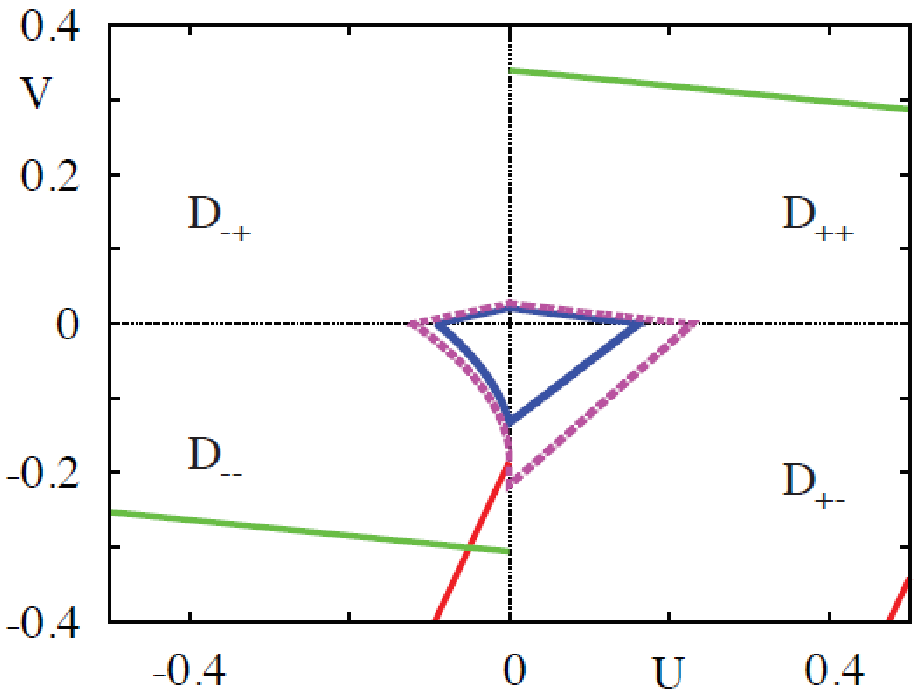
Piecewise linear Wilson-Cowan network

- The expressions may look complex, but because they can be written down in closed form, it can make it easier to perform analysis
- We find a point where just one eigenvalue for the stability problem has left the unit disc
- As before, the corresponding network eigenvector predicts the instability

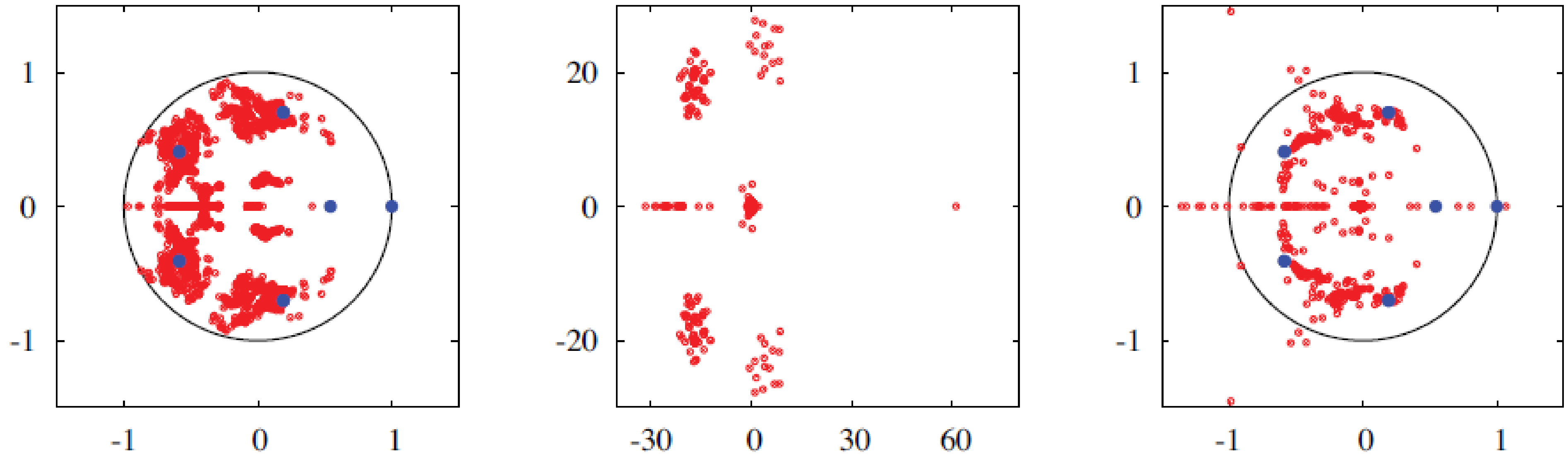




Heaviside Wilson-Cowan network



- Discontinuous vector field, so we use Filippov methods to construct orbits
- (Unstable sliding orbit exists)
- Again, we get some eigenvalues clustering around those of the PWL system



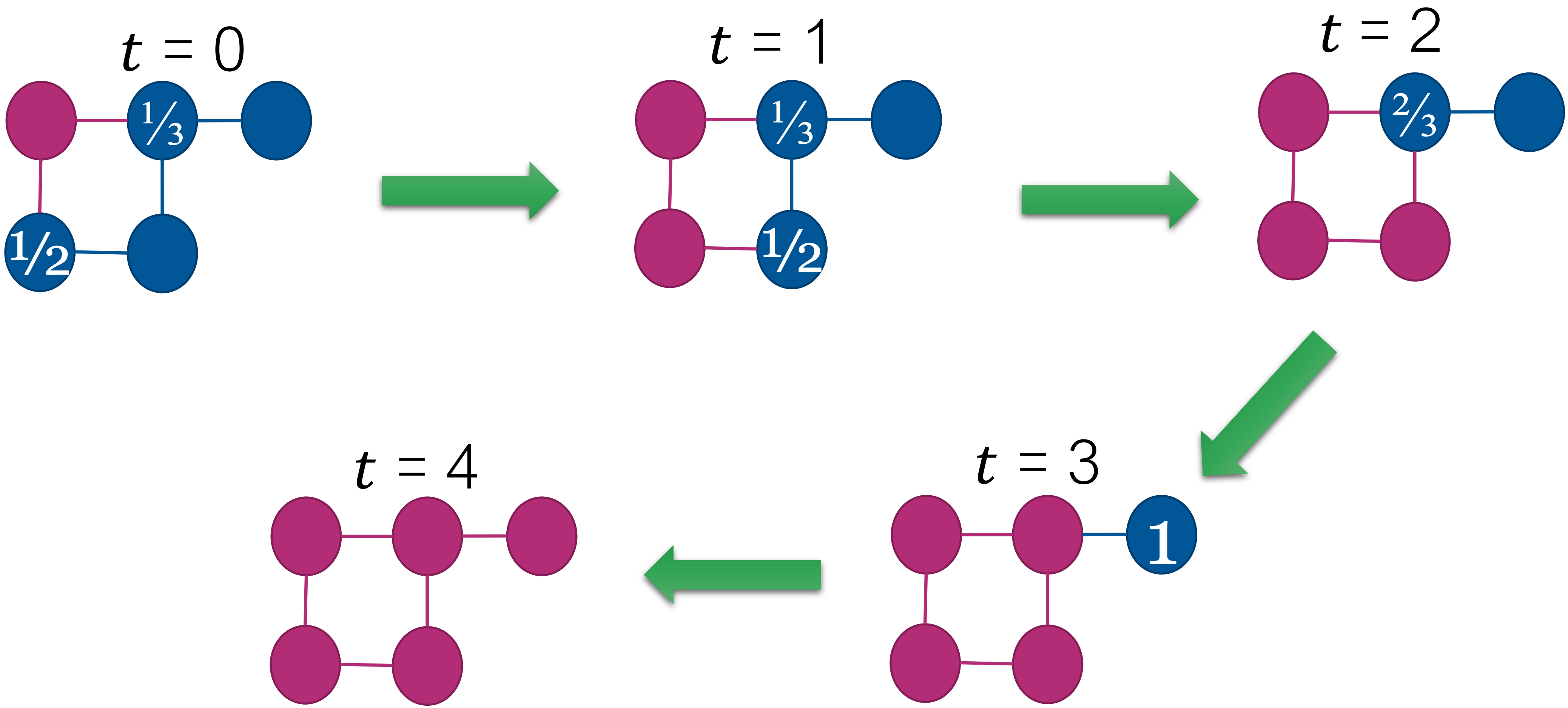
Current/future work – social networks

- Understanding spreading processes on social networks is very important
- 1960s – Models for segregation – Schelling, Axelrod, Granovetter, etc.
- 1978 – "Threshold Models of Collective Behaviour", Granovetter, Am. J. Soc.
- 2002 – "A simple model of global cascades on random networks", Watts, PNAS
  - A node observes the states of its k neighbours, and adopts state 1 if at least a threshold fraction φ of its neighbours are in state 1

Illustration

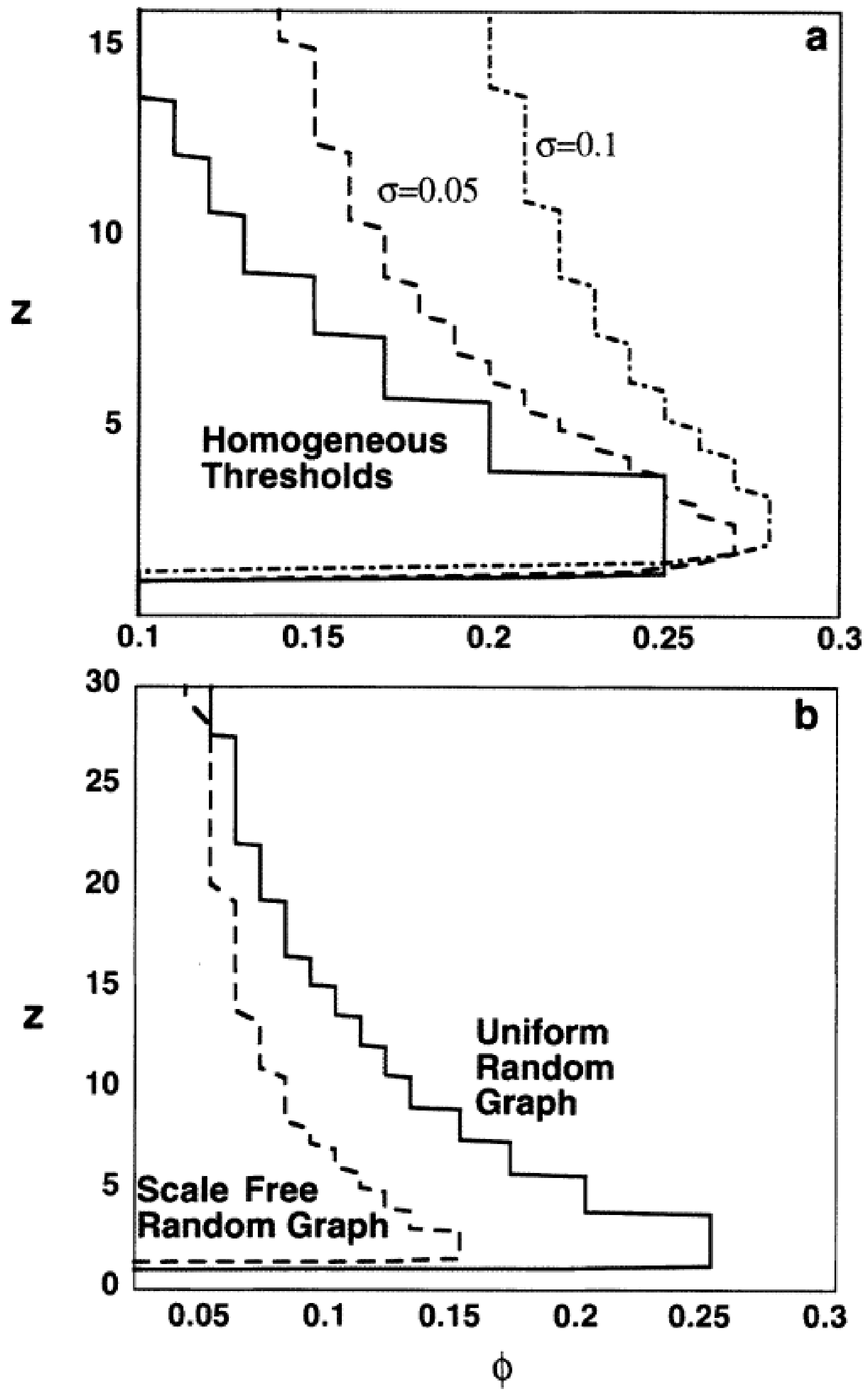
Threshold = 0.4

(same for all nodes in this case)



Watts threshold model

- Social or complex contagion – may require more than one neighbour to spread a behaviour
  - Compare with other types of models, e.g. from epidiemology
- Discrete-time cascades
  - <u>Can we study spatiotemporal structure of cascades?</u>
- Right: Percolation analysis of cascades in the WTM (Watts, 2002)



Other models

Bass "diffusion" model (60s)

$$f(t) = (1 - F(t))(p + qF(t))$$

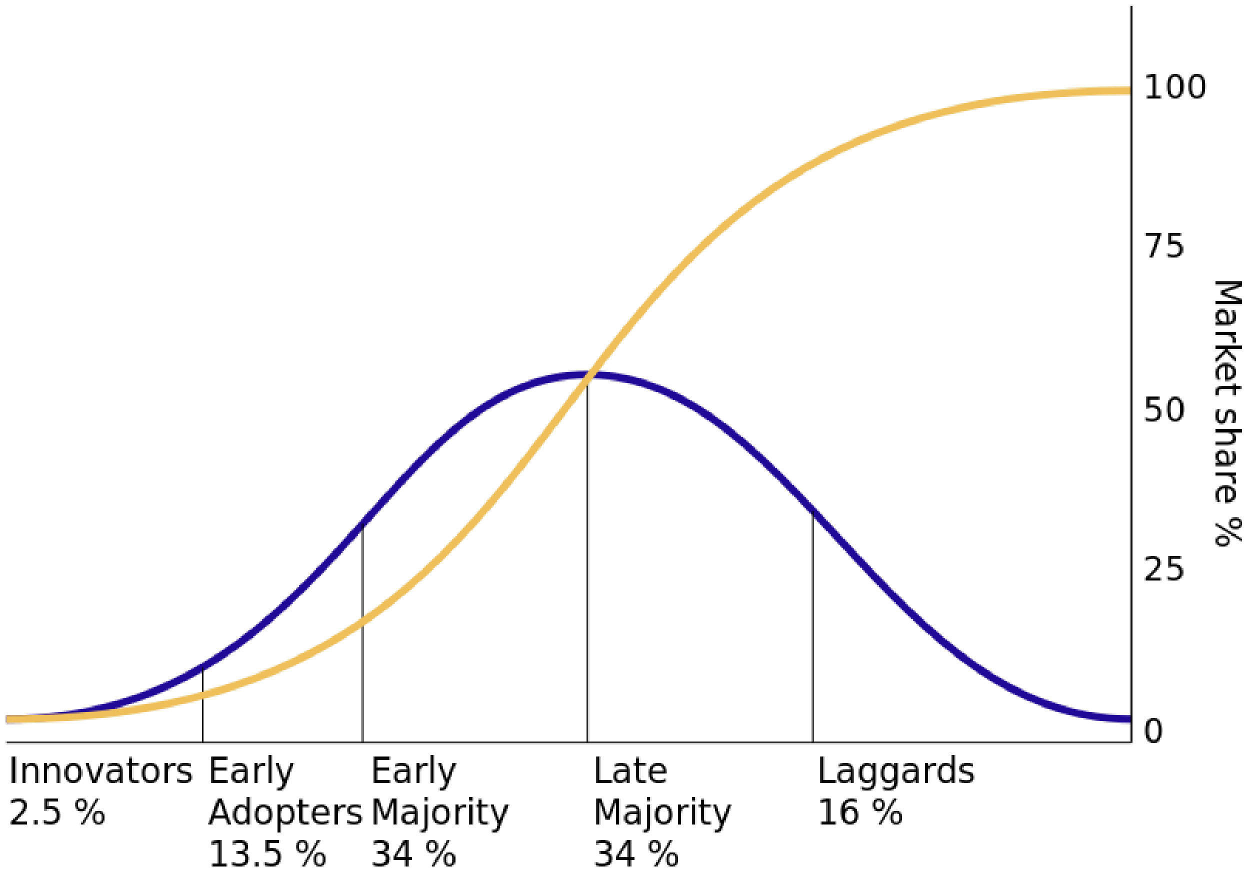


Image credit: Wikipedia "Everett Rogers"

Dynamical-systems threshold model

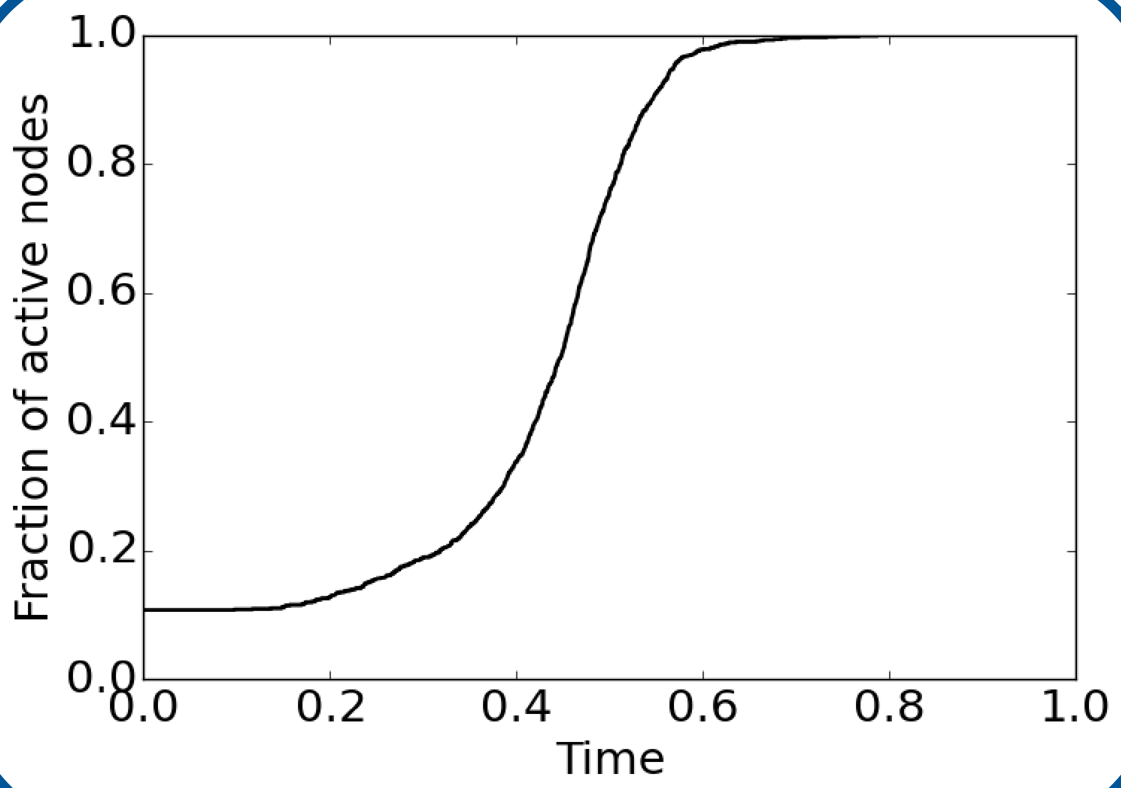
$$\tau_i \frac{dm_i}{dt} = -m_i + \frac{\sum_j p_{ij} \mathbb{1}(m_j \geq R)}{k_{in,i}}$$

- Inspired by integrate-and-fire neurons
- Each node has an internal "motivation" *m*
- $m_i(t) = I[1 - \exp(-\frac{t}{\tau})]$
- Tends to the steady state $I = \frac{\sum_j p_{ij} \mathbb{1}(m_j \geq R)}{k_{in,i}}$
- Once it crosses the threshold *R* it stays active

Dynamical-systems threshold model

Recovers the Watts threshold model when each node moves to its steady state in one step

However, in addition to requiring potentially multiple neighbours, higher numbers of active neighbours also speed up adoption



Sample cascade from an ER $G(n,p)$ graph

For many types of random graph (ER, BA, WS) we observe “S-shaped cascades”

Can we understand why S-shaped curves exist?

We look at Newman—Watts small-world networks

For the simplest case (k=1), this is a ring with shortcuts added uniformly at random

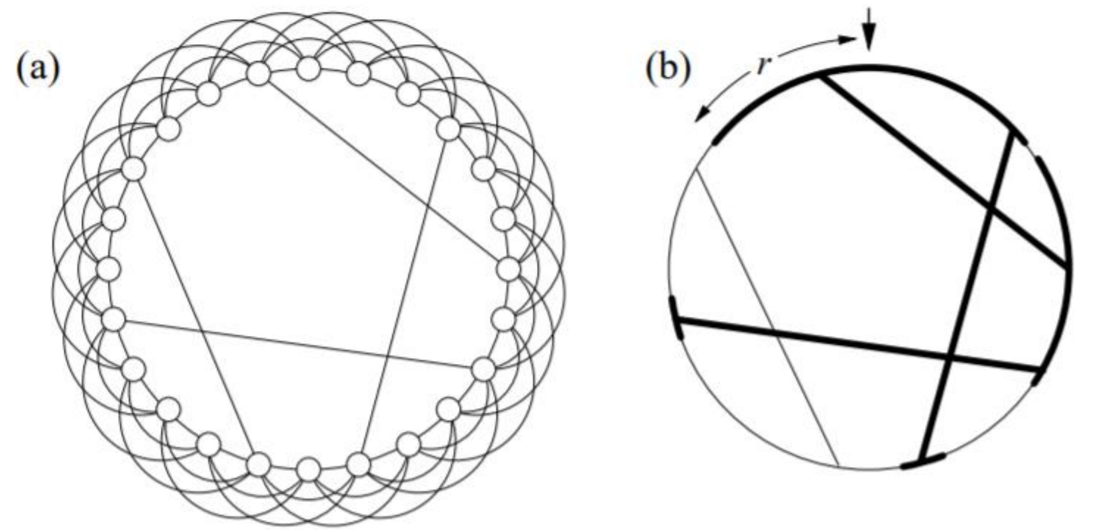


FIGURE 1 (a) A small-world graph of 24 sites with $k = 3$ and four shortcuts. (b) The continuum version of the same graph. The bold lines denote the portion of the graph which is within distance r of the point at the top denoted by the arrow. In this case there are four filled segments, or "clusters", around the perimeter of the graph, or equivalently four gaps between clusters.

Newman & Watts, 1999

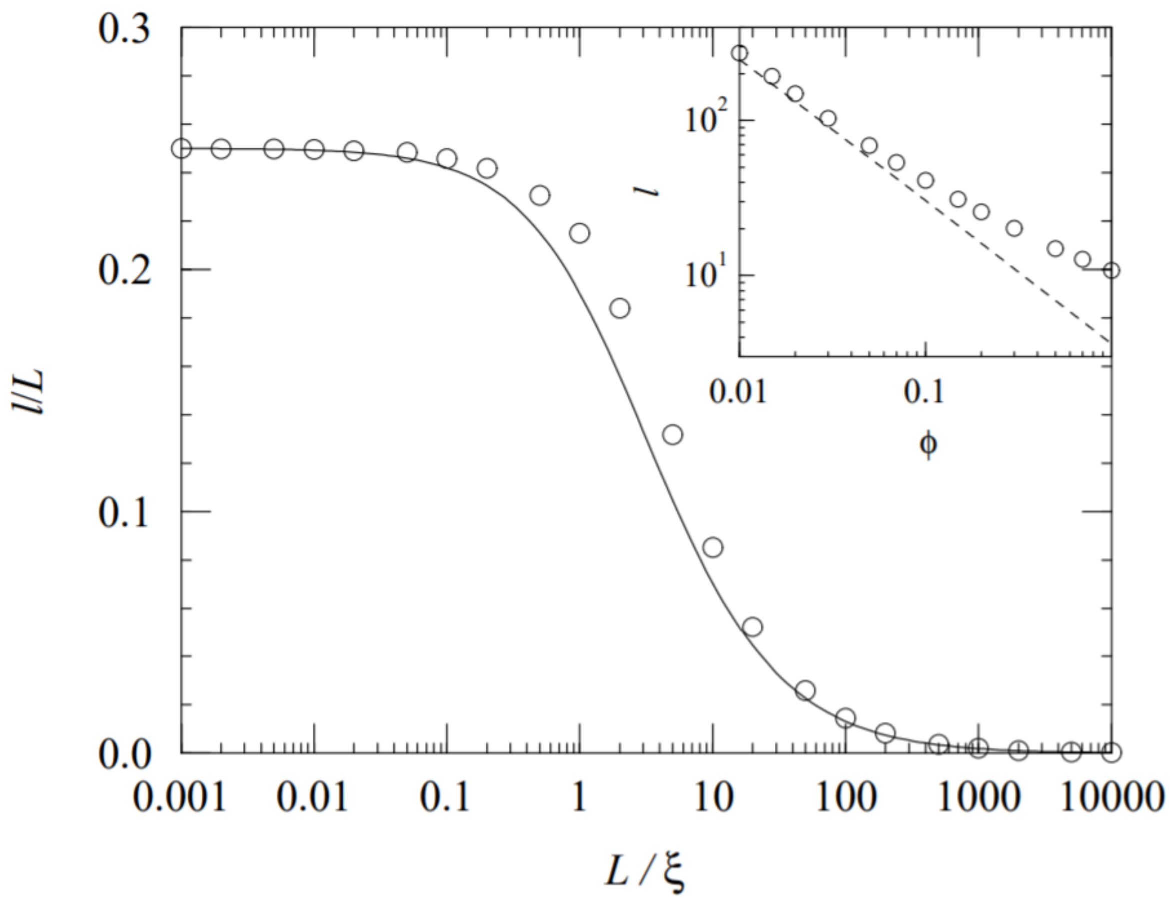
Assume that the cascade spreads approximately at the same speed along paths

We can examine the accuracy of this assumption, since each node only has a few connections

Can we understand why S-shaped curves exist?

Then the spread of cascade calculation is the same as the set of points that can be reached within a certain number of steps

Happily, this calculation has already been done by Newman & Watts (Phys. Rev. E, 1999)



Scaling of mean path length with mean number of shortcuts

Can we understand why S-shaped curves exist?

For the spread of the cascade, we obtain a differential equation for the proportion of active nodes

$$T^* \frac{dF}{dt} = 4k^2 pF(1-F) - \frac{2k}{N}$$

Very close to logistic growth

T^* is our estimator for the time for a node to adopt, based on solving the single-node equation

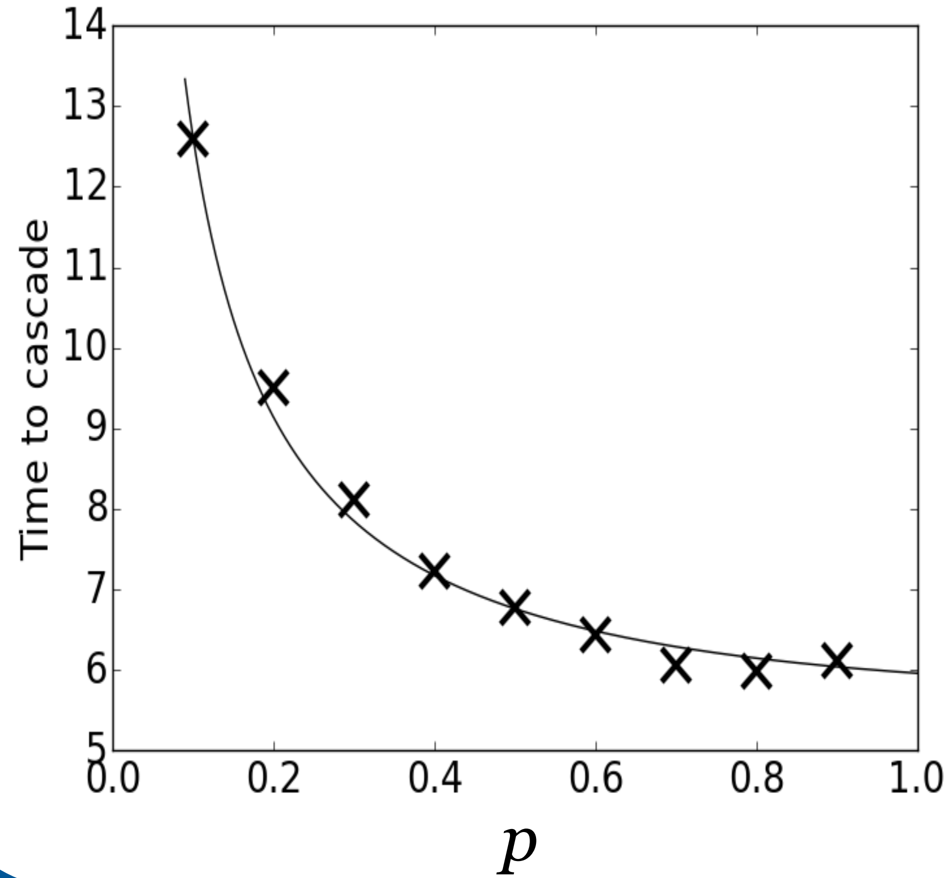
Cascade time as the number of connections increases

Is there a scaling law for the total time for a cascade?

Hypothesis: Cascade time scales with mean path length multiplied by a suitable estimate for the approximate time taken for a node to adopt

For N-W networks, it looks like a good fit

Increasing the number of shortcuts decreases the path length and speeds up cascades



Cascade time as the number of connections increases

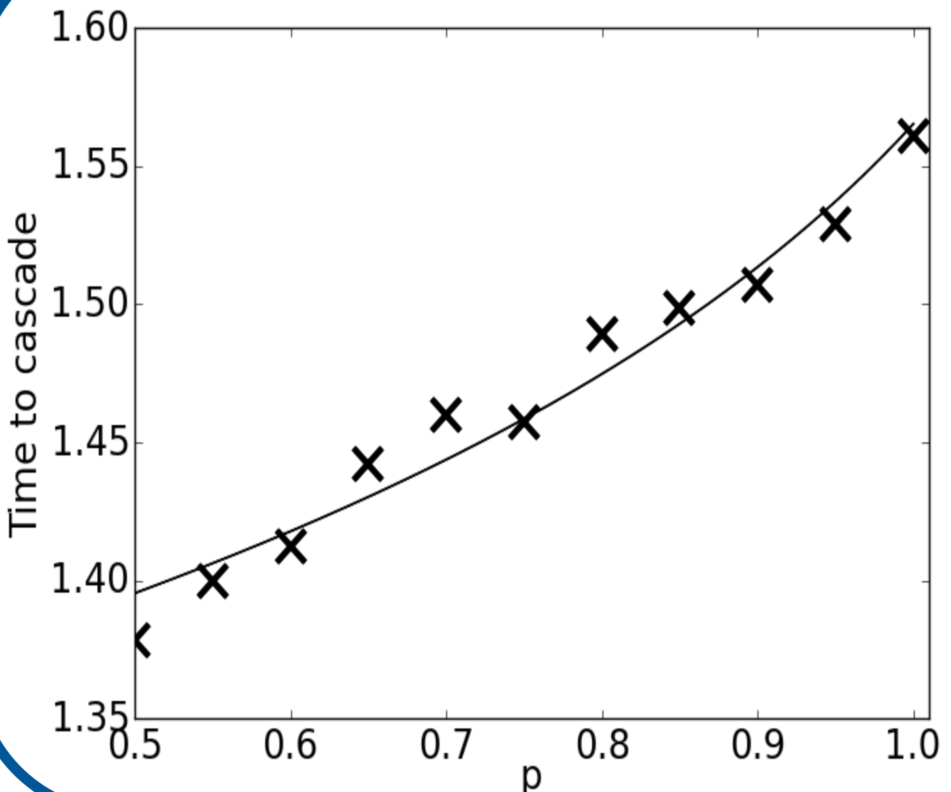
On E-R *G(n,p)* networks with p >> 1/n, mean path length changes slowly - (log n / log pn)

So the dominant scaling term is from time taken to adopt, which goes as log(1-*p*)

The trend supports our hypothesis, but the fit is less good

Need a better estimate for *T**

Compared to the N-W case, many more possible values of $I = \frac{\sum_j p_{ij} \mathbb{1}(m_j \geq R)}{k_{in,i}}$ to consider



Future work

Studying cascades on random regular graphs (joint with Mengdie Yao)

- Much fewer possible sets of states to consider
- For example, d = 3 can only speed up cascades when it has closed triangles
- Mean eccentricity instead of mean shortest path length?

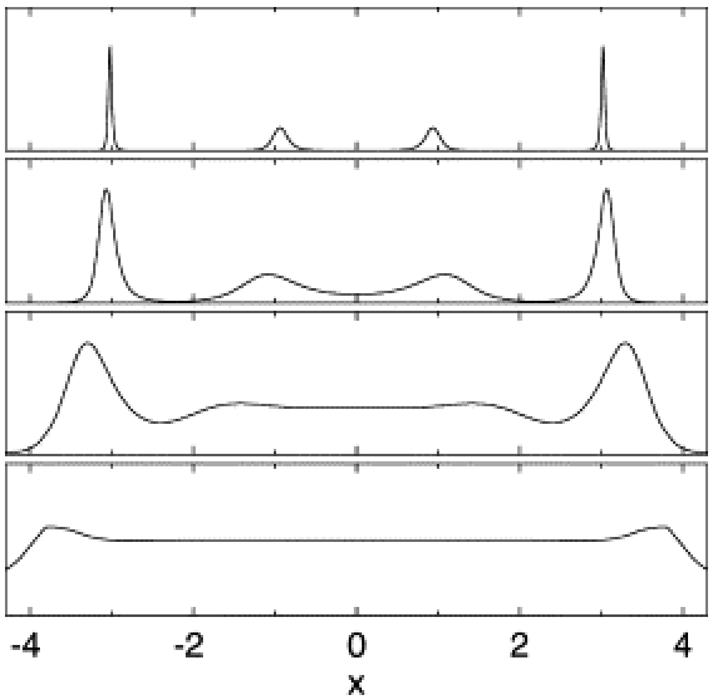
Many extensions to the WTM carry over in a straightforward manner:

- Distributions of thresholds
- “Luddites” or “blocked nodes” that never adopt
- Multistage contagions, where some nodes can attain a higher stage and wield more influence

Future work – bounded confidence models

Bounded confidence models:

- Each node has an opinion in some opinion space (often [-1,1])
- It observes the opinions of it neighbours and moves towards the average of all those within its "opinion radius"
- Can be used to model consensus and echo chambers



Ben-Naim, Krapivsky, Redner, Physica D 2003

Can we get any insight by replacing the "traditional" discrete system with a network dynamical system?

Is there a good mean-field approach?

Adding noise? (We obtained a McKean-Vlasov equation)

Thank you!

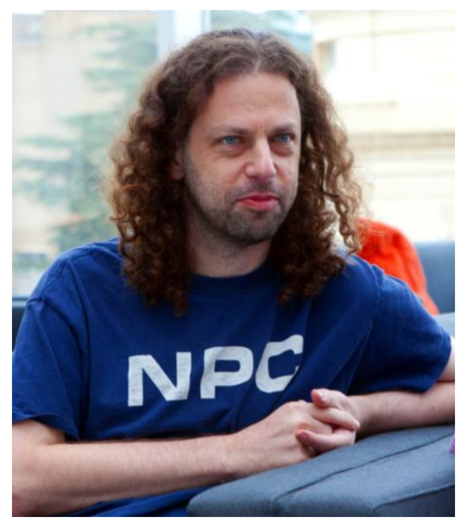
Steve Coombes

Rüdiger Thul

Rachel Nicks

Mustafa Şayli

Joshua Veasy



Mason Porter

(UCLA)

Heather Brooks